1. (10 points)
Consider

$$A = \begin{bmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Compute Null(A), and Col(A). Then find a basis for Null(A), and Col(A), respectively.

A:

In order to evaluate Null(A), we need to solve the equation $A\vec{x} = \vec{0}$.

Perform row reduction

$$\begin{bmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence

$$Null(A) = \text{span}\{\vec{b}\}, \quad \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

The basis is $\{\vec{b}\}$.

The column space is spanned by the pivot columns, with basis given by the 1st, 2nd, 4th columns.

2. (10 points) Consider

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

Find the eigenvalues of A and state their algebraic multiplicities. Then find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

A:

The characteristic polynomial of A is

$$\det(A - \lambda I) = (1-\lambda)(5-\lambda)(4-\lambda) - (-2)(5-\lambda)(-2) = -\lambda(\lambda-5)^2.$$

Hence the eigenvalue are 0 (with multiplicity 1) and 5 (with multiplicity 2).

For the eigenvalue 0, perform row reduction for $A - 0I$ and obtain the normalized vector

$$\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Perform row reduction for $A - 5I$, we obtain an eigenspace spanned by $\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

After normalization, we obtain

$$\vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The basis for $\mathbb{R}^3$ is given by $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

3. (15 points) Solve the initial-value problem

$$y'' - 6y' + 9y = 6te^{3t}, \quad y(0) = 1, \quad y'(0) = 0.$$

A:

Find the roots of the auxiliary equation

$$r^2 - 6r + 9 = (r-3)^2 = 0,$$

so $r = 3$ is the repeated root, and the general solution for the homogeneous equation is

$$y(t) = (C_1 + C_2 t)e^{3t}.$$

Now we find the particular solution. Use method of undetermined coefficients, the particular solution takes the form

$$y_p(t) = t^2(A + Bt)e^{3t}.$$

Plug into the equation and we have

$$y_p(t) = t^3 e^{3t}.$$

The general solution is

$$y(t) = t^3 e^{3t} + (C_1 + C_2 t)e^{3t}.$$

Use the initial condition

$$1 = y(0) = C_1, \quad 0 = y'(0) = 3C_1 + C_2,$$

we have $C_1 = 1, C_2 = -3$.

So the solution is

$$y(t) = e^{3t}(t^3 - 3t + 1).$$

4. (9 points) True or False. If True, explain why. If False, give a counterexample. The correct answer is worth 1 point for each problem. The rest of the points come from the justification.

(a) If the matrix $A \in \mathbb{R}^{3\times 3}$ and A has two rows that are the same, then $\det A = 0$.

A: True. If A has two rows that are the same, then A is not invertible and $\det A = 0$.

(b) Let A be an $n \times n$ matrix. If A^9 is the zero matrix, then the only eigenvalue of A is 0.

A: True. Suppose that $Av = \lambda v$. Then, $A^9 v = \lambda^9 v$ through repeated multiplication. But, $A^9 = 0$, so $A^9 v = 0$. Thus, $\lambda^9 v = 0$. v is an eigenvector and therefore is not the zero vector, so $\lambda^9 = 0$, and $\lambda = 0$. That is, the only possible eigenvalue of A is 0.

(c) There exists a 2×3 matrix A such that $\text{Col}(A) = \{\vec{0}\}$ and $\text{Null}(A) = \{\vec{0}\}$.

A: False. By rank theorem, $\dim\text{Col}(A) + \dim\text{Null}(A) = 3$, so it is impossible that both subspaces have dimension 0.

5. (6 points) $A \in \mathbb{R}^{4\times 4}$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 4, \lambda_4 = 6$, respectively. Calculate the determinant of A. You need to explain how you obtained the answer.

A: A has 4 distinct eigenvalues and hence is diagonalizable as

$$A = VDV^{-1}.$$

Therefore

$$\det A = \det V \det D \det V^{-1} = \det D.$$

The answer is

$$\det A = -1 \cdot 2 \cdot 4 \cdot 6 = -48.$$

6. (10 points) Find the curve $y = C_1 + C_2 2^x$ which gives the best fit (in the least-squares sense) to the three points $(x, y) = (0, 6), (1, 4), (2, 0)$.

A: First write down the equation if the curve indeed passes through all 3 points

$$C_1 + C_2 = 6, \quad C_1 + 2C_2 = 4, \quad C_1 + 4D = 0.$$

The least squares solution of the form $A^T A = A^T b$, with

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}.$$

Compute

$$A^T A = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}, \quad A^T b = \begin{bmatrix} 10 \\ 14 \end{bmatrix}.$$

The solution is

$$C_1 = 8, \quad C_2 = -2.$$

7. (15 points)

(a) Find a solution to the heat equation on a rod of length $L = \pi$

$$\frac{\partial u}{\partial t}(x, t) = 3\frac{\partial^2 u}{\partial x^2}(x, t), \quad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0$$

for all $t > 0$, with the initial condition

$$u(x, 0) = 1 + 3\cos(2x) - 5\cos(3x).$$

A:

The general solution takes the form

$$u(x, t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n e^{-3n^2 t} \cos(nx).$$

Match the initial condition, we find

$$c_0 = 2, \quad c_2 = 3, \quad c_3 = -5,$$

and all other constants vanish.

Hence the solution is

$$u(x,t) = 1 + 3e^{-12t}\cos(2x) - 5e^{-27t}\cos(3x).$$

(b) Consider the function $f(x) = |x|$ defined on the interval $[-1,1]$. Draw a sketch of the function on the interval $[-1,1]$. Find the coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos(n\pi x) + b_n\sin(n\pi x)\right].$$

A:

Sketch.

Since f is an even function, all b_n will vanish. We have

$$a_0 = \int_{-1}^{1} f(x)dx = 1,$$

and

$$a_n = \int_{-1}^{1} f(x)\cos(n\pi x)dx = \frac{2}{\pi^2 n^2}[(-1)^n - 1].$$

8. (10 points)

(a) Let $p(x) = x^2, q(x) = x$, and the inner product for two polynomials $p(x), q(x)$ is defined as

$$\langle p, q\rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Show that

$$\langle p, p\rangle \le \langle p + aq, p + aq\rangle$$

for any $a \in \mathbb{R}$.

A:

First evaluate that $\langle p, q\rangle = 0$. Hence p, q are orthogonal polynomials under this inner product.

Then

$$\langle p + aq, p + aq\rangle = \langle p, p\rangle + 2a\langle p, q\rangle + a^2\langle p, q\rangle = \langle p, p\rangle + a^2\langle q, q\rangle \ge \langle p, p\rangle.$$

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, which defines an inner product on $\mathbb{R}^2$ as follows

$$\langle \vec{x}, \vec{y}\rangle = \vec{x}^T A\vec{y}, \quad \vec{x}, \vec{y} \in \mathbb{R}^2.$$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Use the Gram-Schmidt process to find a vector that is orthogonal to $\vec{v}_1$ under this inner product. (You DO NOT need to prove that $\vec{x}^T A\vec{y}$ is indeed an inner product)

A:

Take for example $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Compute

$$\langle \vec{v}_1, \vec{v}_1 \rangle = 2, \quad \langle \vec{v}_1, \vec{v}_2 \rangle = 1.$$

The second vector is

$$\vec{w}_2 = \vec{v}_2 - \vec{v}_1 \frac{\langle \vec{v}_1, \vec{v}_2 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}.$$

9. (10 points) The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ orthogonally projects every point in $\mathbb{R}^3$ onto the plane $x + y = 0$. Write down the matrix representation of T in the standard basis of $\mathbb{R}^3$.

A: First, the solution to the linear system $x + y = 0$ is the subspace

$$W = \text{span} \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

W has an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$

$$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then linear transformation T is then defined as

$$T(\vec{v}) = \vec{v}_1(\vec{v}_1 \cdot \vec{v}) + \vec{v}_2(\vec{v}_2 \cdot \vec{v}).$$

Direct computation shows that

$$T(\vec{e}_1) = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \quad T(\vec{e}_3) = \vec{e}_3.$$

Hence the matrix representation of T in the standard basis is

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1. \end{bmatrix}$$