Math 54. Final exam. Fall 2017

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This is a closed book, closed notes exam. You need to justify every one of your answers unless you are asked not to do so. Completely correct answers given without justification will receive little credit. Look over the whole exam to find problems that you can do quickly. Hand in this exam before you leave.

DO NOT tear out any page or add any page. This is crucial for the grading process with gradescope. Write your name on the top left corner of each page. If your answer appears in the scratch paper appended in the end, refer to your answer using the page number.

Your name: ________________________________

Your SID : ________________________________

Good luck, and have a nice break!

1. (10 points) Consider

$$A = \begin{bmatrix} 1 & 3 & 5 & 9 \\ 2 & 4 & 6 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

Compute Null(A), and Col(A).
Then find a basis for Null(A), and Col(A), respectively.

2. (10 points) Consider

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & 0 \\ -2 & 0 & 4 \end{bmatrix}.$$

Find the eigenvalues of A and state their algebraic multiplicities. Then find an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A.

3. (15 points) Solve the initial-value problem

$$y'' - 6y' + 9y = 6te^{3t}, \quad y(0) = 1, \quad y'(0) = 0.$$

4. (9 points) True or False. If True, explain why. If False, give a counterexample. The correct answer is worth 1 point for each problem. The rest of the points come from the justification.

(a) If the matrix $A \in \mathbb{R}^{3\times 3}$ and A has two rows that are the same, then $\det A = 0$.

(b) Let A be an $n \times n$ matrix. If A^9 is the zero matrix, then the only eigenvalue of A is 0.

(c) There exists a 2×3 matrix A such that $\mathrm{Col}(A) = \{\vec{0}\}$ and $\mathrm{Null}(A) = \{\vec{0}\}$.

5. (6 points) $A \in \mathbb{R}^{4\times 4}$ has eigenvalues $\lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 4, \lambda_4 = 6$, respectively. Calculate the determinant of A. You need to explain how you obtained the answer.

6. (10 points) Find the curve $y = C_1 + C_2 2^x$ which gives the best fit (in the least-squares sense) to the three points $(x, y) = (0, 6), (1, 4), (2, 0)$.

7. (15 points)
(a) Find a solution to the heat equation on a rod of length $L = \pi$

$$\frac{\partial u}{\partial t}(x,t) = 3\frac{\partial^2 u}{\partial x^2}(x,t), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$$

for all $t > 0$, with the initial condition

$$u(x,0) = 1 + 3\cos(2x) - 5\cos(3x).$$

(b) Consider the function $f(x) = |x|$ defined on the interval $[-1, 1]$. Draw a sketch of the function on the interval $[-1, 1]$. Find the coefficients a_n, b_n such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\pi x) + b_n \sin(n\pi x)\right].$$

8. (10 points)

(a) Let $p(x) = x^2, q(x) = x$, and the inner product for two polynomials $p(x), q(x)$ is defined as

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)dx.$$

Show that

$$\langle p, p \rangle \leq \langle p + aq, p + aq \rangle$$

for any $a \in \mathbb{R}$.

(b) Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, which defines an inner product on $\mathbb{R}^2$ as follows

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T A \vec{y}, \quad \vec{x}, \vec{y} \in \mathbb{R}^2.$$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Use the Gram-Schmidt process to find a vector that is orthogonal to $\vec{v}_1$ under this inner product. (You DO NOT need to prove that $\vec{x}^T A \vec{y}$ is indeed an inner product)

9. (10 points) The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ orthogonally projects every point in $\mathbb{R}^3$ onto the plane $x + y = 0$. Write down the matrix representation of T in the standard basis of $\mathbb{R}^3$.

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