Second Midterm Solutions

Place all answers on the question sheet provided. The exam is closed textbook/notes/homework, but you may bring one two-sided cheat sheet. You are allowed to use a calculator, but not a computer. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument. This exam has a total of 100 points.

First Name: ____________________

Last Name: ____________________

1(a)	1(b)	2(a)	2(b)	3	4(a)	4(b)	4(c)	Total

Honor Code

I resolve

i) not to give or receive aid during this examination, and

ii) to take an active part in seeing that other students uphold this Honor Code.

Signature: ____________________

1. The number of kids who "Trick or Treat" in your street depends on the weather on Halloween. In particular, we assume that if the weather is bad, the number of "trick or treaters" follows a Poisson distribution with mean 5, while if the weather is good, the number of "trick or treaters" follows a Poisson distribution with mean 15. The weather on Halloween on any given year is either good or bad with probabilities 0.7 and 0.3, respectively.

   (a) (15 PTS) Compute the probability that next year there will be 2 or more "trick or treaters" on your street.

   Solution:

   Let A denote the event that the weather on Halloween is good; $P(A) = 0.7$. Let N denote the number of kids who "Trick or Treat" on Halloween. Then, the problem states that the conditional PMF of Y given A is

$$p_{N|A}(n) = P(N = n|A) = \frac{e^{-15}(15)^n}{n!}, \qquad n = 0, 1, 2, \ldots,$$

   and the conditional PMF of Y given A^c is

$$p_{N|A^c}(n) = P(N = n|A^c) = \frac{e^{-5}5^n}{n!}, \qquad n = 0, 1, 2, \ldots.$$

   Then,

$$\begin{aligned} P(N \geq 2) &= P(N \geq 2|A)P(A) + P(N \geq 2|A^c)P(A^c) \\ &= \left(1 - e^{-15} - e^{-15}(15)\right) P(A) + \left(1 - e^{-5} - e^{-5}5\right) P(A^c) \\ &= \left(1 - e^{-15} - e^{-15}(15)\right) (0.7) + \left(1 - e^{-5} - e^{-5}5\right) (0.3) \end{aligned}$$

(b) (10 PTS) This year you were out of town for Halloween. Upon returning home, your neighbors told you that they saw no kids on your street on Halloween. What is the probability that the weather was good?

Solution:

We want to compute $P(A|N=0)$, so we use Bayes' rule to obtain

$$P(A|N=0) = \frac{P(N=0,A)}{P(N=0)} = \frac{P(N=0|A)P(A)}{P(N=0|A)P(A)+P(N=0|A^c)P(A^c)}$$
$$= \frac{e^{-15}(0.7)}{e^{-15}(0.7)+e^{-5}(0.3)} = \frac{1}{1+e^{10}(3/7)}$$

2. The revenue obtained by a popular clothes store comes from two sources: in-store purchases and online purchases. Let S denote the revenue from in-store purchases and let T denote the revenue from online purchases, both measured in millions of dollars. We assume that S and T are independent exponential random variables with means 2 million dollars and 1 million dollars, respectively.

   (a) (10 PTS) Compute the mean and the variance of the store's total revenue.

   Solution:

   Since both S and T are exponentially distributed, their parameters are $\lambda = 1/2$ and $\mu = 1$, respectively. It follows that

$$E[S+T] = 2 + 1 = 3 \quad \text{millions},$$
$$\text{var}(S+T) = \text{var}(S) + \text{var}(T) = (2)^2 + (1)^2 = 5.$$

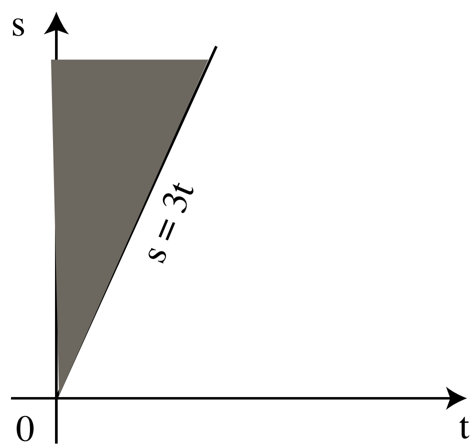
(b) (10 PTS) Set up a double integral that when computed gives the probability $P(S > 3T)$; make sure all the limits of integration are clearly specified. Do not solve it.

Solution:

Since S and T are independent, their joint PDF is given by

$$f_{S,T}(s,t) = \frac{1}{2}e^{-s/2}e^{-t}, \qquad s,t \geq 0.$$

To compute the probability we seek note that we need to integrate the shaded area below:



It follows that

$$P(S > 3T) = \int_0^\infty \int_{3t}^\infty \frac{1}{2}e^{-s/2}e^{-t}\,ds\,dt = \int_0^\infty \int_0^{s/3} \frac{1}{2}e^{-s/2}e^{-t}\,dt\,ds.$$

3. (25 PTS) Determine whether the following statements are True or False.

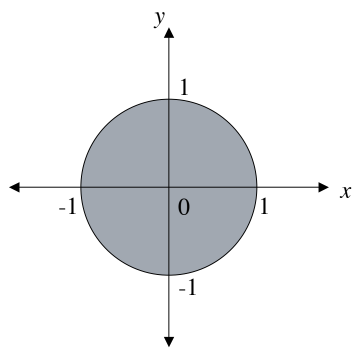
__________ The function $f_Y(y) = 8y^{-3}$, $y \geq 2$, is a legitimate PDF.

__________ The continuous random variables having joint density $f_{X,Y}(x, y) = e^{-(x+y)}$, $x \geq 0$, $y \geq 0$, are independent.

__________ The function $f_Z(z) = 3z^2 + 1$, $0 \leq z \leq 1$ is a legitimate PDF.

__________ The function $f_W(w) = 5 - 3w$, $0 \leq w \leq 2$, is a legitimate PDF.

__________ If the joint density of X and Y is nonzero on the shaded area below, they could be independent.



Solutions:

(i) True, because $f_Y(y)$ is nonnegative and integrates to 1.

(ii) True, because if we let $f_X(x) = e^{-x}$ and $f_Y(y) = e^{-y}$ for $x, y \geq 0$, then $f_{X,Y}(x, y) = f_X(x)f_Y(y)$.

(iii) False, because $f_Z(z)$ integrates to 2 over $[0, 1]$.

(iv) False, because $f_W(w)$ is negative for $5/3 \leq w \leq 3$.

(v) False, because the region where X and Y are defined is $x^2 + y^2 \leq 1$, which cannot be split into two ranges, one for x and one for y, that do not depend on each other.

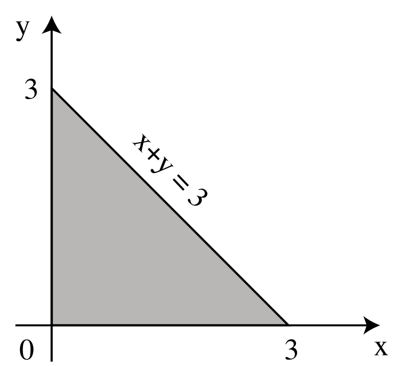
4. A Berkeley student has started studying for an exam that will take place in 3 days. However, she also has a homework for another class that is due on that same day. Let X denote the amount of time she spends studying for her exam, and let Y denote the amount of time that she spends working on her homework for the other class, both measured in days. Note that $X + Y \leq 3$ since both the exam and the homework will be over in 3 days.

   Suppose that the joint PDF of X and Y is given by

$$f_{X,Y}(x, y) = \frac{2x}{9}, \qquad 0 \leq x \leq 3, \quad 0 \leq y \leq 3 - x.$$

   (a) (5 PTS) Plot the region of the (x, y)-plane where $f_{X,Y}$ is non-zero.

   Solution:



   (b) (10 PTS) Compute the marginal PDF of X. Do not forget to specify the range over which it is non-zero.

   Solution:

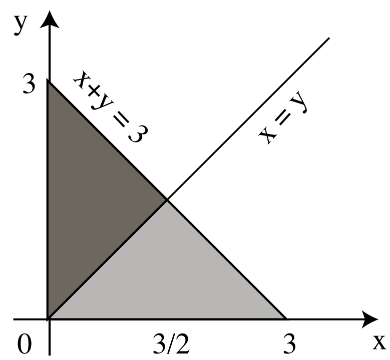
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_0^{3-x} \frac{2x}{9}\, dy = \frac{2x}{9} \int_0^{3-x} dy = \frac{2x(3-x)}{9}$$

   for $0 \leq x \leq 3$.

(c) (15 PTS) Compute the probability that she ends up spending more time on her homework than studying.

Solution:

We need to integrate the darker area below:



$$
\begin{aligned}
P(Y > X) &= \int_0^{3/2} \int_x^{3-x} f_{X,Y}(x,y)\,dy\,dx = \int_0^{3/2} \int_x^{3-x} \frac{2x}{9}\,dy\,dx \\
&= \int_0^{3/2} \frac{2x}{9}(3-2x)\,dx = \frac{2}{9}\int_0^{3/2} (3x - 2x^2)\,dx \\
&= \frac{2}{9}\left(\frac{3x^2}{2} - \frac{2x^3}{3}\right)\Bigg|_0^{3/2} \\
&= \frac{2}{9}\left(\frac{27}{8} - \frac{18}{8}\right) = \frac{1}{4}
\end{aligned}
$$