First Midterm

Place all answers on the question sheet provided. The exam is closed textbook/notes/homework, but you may bring one two-sided cheat sheet. You are allowed to use a calculator, but not a computer. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument. This exam has a total of 100 points.

First Name: ______________________

Last Name: ______________________

1(a)	1(b)	2(a)	2(b)(i)	2(b)(ii)	2(c)	3(a)	3(b)	3(c)	Total

Honor Code

I resolve

i) not to give or receive aid during this examination, and

ii) to take an active part in seeing that other students uphold this Honor Code.

Signature: ______________________

1. Mrs. Simpson collects issues of TIME magazine. She has exactly 30 magazines piled up right now, all different from each other. We assume that every possible ordering of the magazines is equally likely.

   (a) (10 PTS) Compute the probability that the December 1st issue, one of the 30 she has, is the one at the bottom of the pile.

   Solution:

   There are 29! permutations that have the December 1st issue last, and a total of 30! permutations, therefore, the probability is

   $$\frac{29!}{30!} = \frac{1}{30}$$

   In other words, the fact that we're asking for the last position is the same as any other position, for example the first one.

   (b) (15 PTS) Of the 30 issues, 8 have something related to the 2016 US election on the cover. Mrs. Simpson picks the top 5 magazines and the sixth one has a cover related to the election. Given that the sixth one has a cover related to the election, what is the probability that none of the other other five do?

   Solution:

   Let A denote the event that the sixth issue has a cover related to the 2016 election, and let B denote the probability that the first 5 issues do not have something related to the election. The question is asking for $P(B|A)$.

   Now compute

   $$P(A \cap B) = \frac{22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 8}{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25}$$

   and by choosing one of the 8 election issues to be in the 6th position and permuting the remaining 29, we obtain that

   $$P(A) = \frac{8 \cdot 29!}{30!} = \frac{8}{30}$$

   Hence,

   $$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{22 \cdot 21 \cdot 20 \cdot 19 \cdot 18 \cdot 8}{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26 \cdot 25} \cdot \frac{30}{8} = \frac{22 \cdot 19}{29 \cdot 13 \cdot 5} = 0.2217$$

2. During the summer, Tom works delivering the local newspaper around his neighborhood; the newspaper comes out every week. Every Monday, Tom will stop by the depot to pick up a bulk of newspapers and put them in a cart, then he'll start walking around his assigned streets delivering one newspaper per house. However, Tom does not usually count how many newspapers he picks up from the depot, so sometimes he either has leftover newspapers after finishing his route or he runs out of newspapers before getting to the last house in his route.

   Suppose that his route consists of 20 houses. Define X to be the number of newspaper he picks up on a given Monday. The PMF of X is given below:

x	17	18	19	20	21
$p_X(x)$	0.1	0.2	0.3	0.3	0.1

   (a) (10 PTS) Compute the probability that one or more houses do not receive a newspaper that day.

   Solution:

   The question asks for $P(X \leq 19)$, which can be computed to be

$$P(X \leq 19) = P(X = 19) + P(X = 18) + P(X = 17) = 0.1 + 0.2 + 0.3 = 0.6$$

(b) Let Y denote the number of houses that do not receive a newspaper that day.

i. (15 PTS) Compute the PMF of Y.
Solution:
Note that $Y = \max\{20 - X, 0\}$, and since the minimum number of newspapers Tom will deliver is 17, then the possible values for Y are $\{0, 1, 2, 3\}$. The probabilities of each of these values are given by:

$$P(Y = 0) = P(X \geq 20) = P(X = 20) + P(X = 21) = 0.3 + 0.1 = 0.4,$$

$$P(Y = 1) = P(X = 19) = 0.3, \qquad P(Y = 2) = P(X = 18) = 0.2,$$
$$P(Y = 3) = P(X = 17) = 0.1$$

Equivalently,

$$p_Y(0) = 0.4, \quad p_Y(1) = 0.3, \quad p_Y(2) = 0.2, \quad p_Y(3) = 0.1$$

ii. (10 PTS) Compute the mean of Y.
Solution:
Using our answer from (b)(ii) we get:

$$\begin{aligned} E[Y] &= 0 \cdot p_Y(0) + 1 \cdot p_Y(1) + 2 \cdot p_Y(2) + 3 \cdot p_Y(3) \\ &= 0.3 + 2(0.2) + 3(0.1) \\ &= 1 \end{aligned}$$

(c) (10 PTS) During the next summer, Tom will be delivering newspapers along his route a total of 8 times. Suppose your house is the last one in Tom's route. Compute the probability that you get exactly 5 newspapers over this 8-week period.

Solution:

Each of the 8 weeks, you will independently receive a newspaper with probability

$$p = P(Y = 0) = 0.4$$

Therefore, the probability that you receive exactly 5 out of the 8 newspapers is

$$\binom{8}{5} p^5 (1-p)^{8-5} = \binom{8}{5} (0.4)^5 (0.6)^3 = 56\,(0.4)^5 (0.6)^3 = 0.1239$$

3. In a certain video game there is a special quest in which the player needs to get through a maze. The maze consists of up to three stages. Stages 1 and 2 have each three doors, of which only one leads to the next stage; stage three has only 2 doors, both of which lead to the exit, but one of which gives you an extra reward. Every time the player chooses the wrong door in stages 1 and 2, he/she gets sent back to the entrance of that same stage. The player is allowed to choose the wrong door once, and will be directly sent to the exit on the second incorrect choice.

   Since each stage pays a reward of 1 point (paid only upon entering a stage for the first time); choosing the correct door in stage 3 gives an additional 1 point. The goal is to collect as many points as possible. .

   Assume that the game chooses the identity of the correct door in each stage with equal probability, and independently of previous stages.

   (a) (8 pts) Compute the probability that you complete stage 1 after having made exactly one mistake; i.e., that you pick the correct door on your second choice. Do it both under the assumption that the player keeps track of the doors they have chosen and under the assumption that they do not.

   Solution:

   There are three choices, all equally likely. Under the assumption that you keep track of the doors you have chosen, the probability is

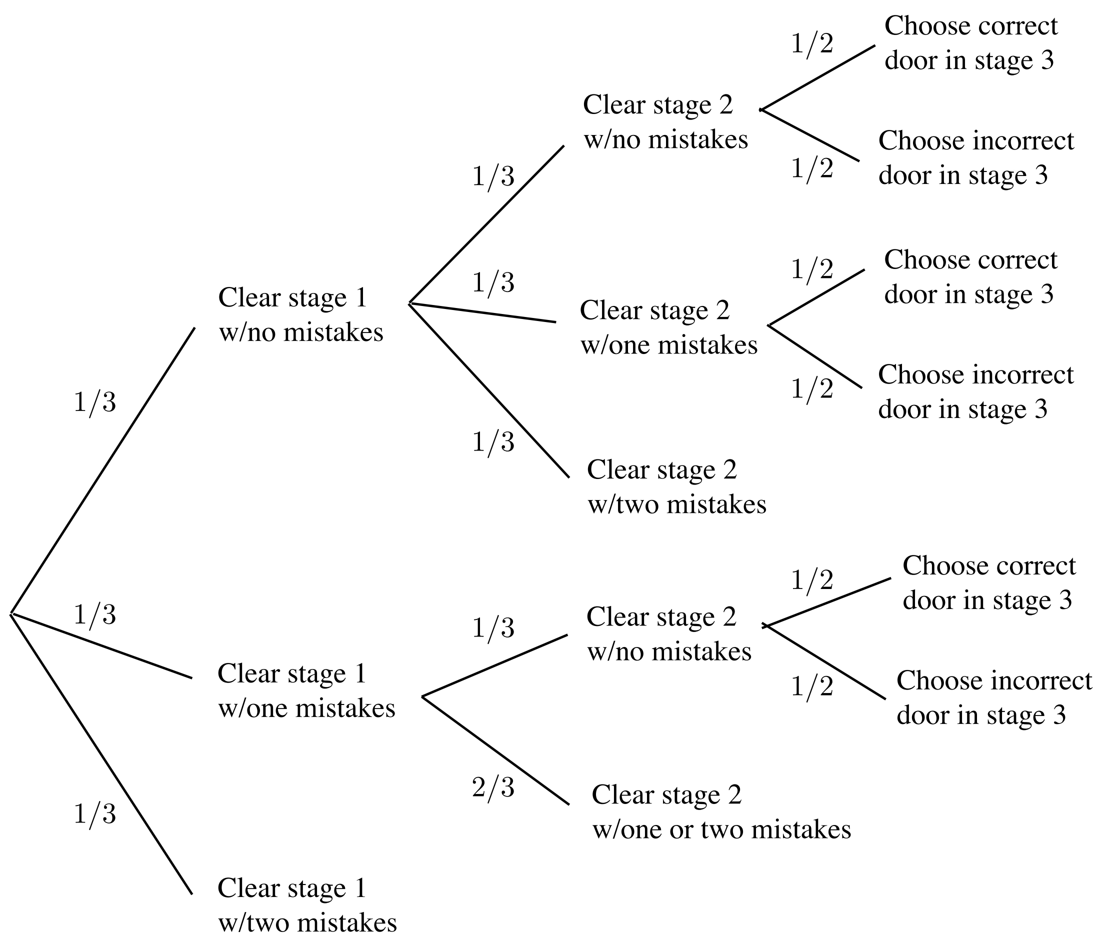
$$\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

   If you do not keep track of the doors you have chosen, the probability is

$$\frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}$$

(b) (15 pts) The following tree indicates all the possible runs of the maze. Complete all the branch probabilities under the assumption that the player keeps track of which doors he/she has chosen in each stage.

Solution:



(c) (10 pts) Compute the PMF of the random variable X = number of points obtained from one run of the maze, under the assumption that the player keeps track of which doors he/she has chosen in each stage. Note that X can take the values $\{1, 2, 3, 4\}$.

Solution:

Note that $X = 4$ occurs when the player makes at most one mistake in the first two stages together, and then gets the third stage correctly, therefore

$$P(X = 4) = \frac{1}{3} \cdot \frac{1}{3} \cdot 12 + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{3}{18} = \frac{1}{6}$$

For $X = 3$ the player needs to make at most one mistake in the first two stages together, but then needs to get the third stage door wrong, which occurs with probability

$$P(X = 3) = \frac{1}{3} \cdot \frac{1}{3} \cdot 12 + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{3}{18} = \frac{1}{6}$$

For $X = 2$ the player must either make two mistakes in stage 2 and none in stage 1, or one mistake in stage 2 and 1 or more mistakes in stage 2, which occurs with probability

$$P(X = 2) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{1}{3}$$

Finally, $X = 1$ occurs only if the player makes to mistakes in the first stage, which occurs with probability

$$P(X = 1) = \frac{1}{3}$$