Math 110 (Fall 2018) Midterm I (Monday September 24, 12:10-1:00)

1. Mark each of the following (1)–(4) True (T) or False (F). Don't give a full proof but provide a brief justification, no more than a few sentences. (Correct with justification = 4 pts, Correct but no or wrong justification = 2 pts, Incorrect answer = 0 pt.) See the front page for notation.

(1) (F) Let V be a finite-dimensional vector space, β a basis for V, and W a subspace of V. Then there is a subset of β which is a basis for W.

For example, if $V = \mathbb{R}^2$, $\beta = \{(1,0),(0,1)\}$, and $W = \{(a,a) : a \in \mathbb{R}\}$, then neither of the vectors in β is in W, so no subset of β is a basis for W.

(2) (F) Let $T : V \to W$ and $U : W \to Z$ be linear maps of finite dimensional vector spaces. If UT is invertible, then either U or T (or possibly both) is invertible.

[A silly example] Take $V = Z = \{0\}$, W any nonzero space, thus T, U are the zero maps. Then UT is invertible, but clearly U, T are not (since $\dim W$ is unequal to $\dim V$ and $\dim Z$).

[A less silly example] Take $T : \mathbb{R} \to \mathbb{R}^2$, $T(x) = (x, 0)$ and $U : \mathbb{R}^2 \to \mathbb{R}$ with $U(x,y) = x$. Then $UT(x) = x$ so UT is invertible, but U and T are not (again for the dimension reason).

(3) (F) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ (over $\mathbb{R}$) such that $R(T) = N(T)$.

The dimension theorem says $\text{rank}(T) + \text{nullity}(T) = 3$ so rank and nullity must be different. Therefore $R(T) \neq N(T)$. (Namely if $R(T)$ and $N(T)$ were equal, their dimensions should be equal too.)

(4) (T) Let $T : V \to W$ be a linear map of finite dimensional vector spaces. If $T^t : W^* \to V^*$ is the zero transformation then T is also the zero transformation.

[Approach 1: more standard] By the dictionary we built up in chapter 2, the statement translates into "If A^t is the zero matrix then so is A." and this is true (since A^t simply rearranges the entries of A). Such an answer is deemed correct.

For a rigorous way, let β, γ be ordered bases for V, W, and β^*, γ^* the dual bases. Then

$$T^t = 0 \;\Rightarrow\; [T^t]_{\gamma^*}^{\beta^*} = 0 \;\Rightarrow\; [T]_\beta^\gamma = 0 \;\Rightarrow\; T = 0.$$

(The middle arrow comes from Theorem 2.25. We're writing 0 for the zero map or the zero matrix.)

[Approach 2: using double duality] This is more complicated but a basis-free approach.

$$T^t = 0 \Rightarrow \forall g \in W^*,\, T^t(g) = 0 \Rightarrow \forall g \in W^*,\, g(T) = 0 \Rightarrow \forall g \in W^*, v \in V,\, (g(T))(v) = g(T(v)) = 0.$$

$$\Rightarrow \forall g, v,\, \widehat{T(v)}(g) = g(T(v)) = 0 \Rightarrow \forall v,\, \widehat{T(v)} = 0 \Rightarrow \forall v,\, T(v) = 0 \Rightarrow T = 0.$$

(Here $\hat{x} \in V^{**}$ denotes the element as on p.122. The implication $\widehat{T(v)} = 0 \;\Rightarrow\; T(v) = 0$ follows from the double duality isomorphism, Theorem 2.26.)

2. (16 pts) Let $T : V \to W$ be an isomorphism, where V, W are finite dimensional vector spaces. If β is a basis for V, prove that $T(\beta) = \{T(x) : x \in \beta\}$ is a basis for W.

Let $n = \dim V$. Since T is an isomorphism, $\dim W = \dim V = n$. (Lemma, §2.4, p.101.) Since $|\beta| = n$ and T is 1-1, we have $|T(\beta)| = n$, which is $\dim W$. So it's enough to show $T(\beta)$ generates W. But we know

$$R(T) = \text{span}(T(\beta))$$

(Theorem 2.2) while $R(T) = W$ as T is onto. Hence $W = \text{span}(T(\beta))$ and we're done. $\square$

For a different approach (knowing $|T(\beta)| = n$ as above), it's also enough to show that $T(\beta) = \{T(v_1), ..., T(v_n)\}$ is linearly independent, where $\beta = \{v_1, ..., v_n\}$. So assume $\sum_{i=1}^n a_i T(v_i) = 0$ with a_i scalars. Then $T(\sum_{i=1}^n a_i v_i) = 0$, so $\sum_{i=1}^n a_i v_i \in N(T)$. Since T is 1-1, this implies $\sum_{i=1}^n a_i v_i = 0$. By linear independence of β, all the a_i are zero. Therefore $\{T(v_1), ..., T(v_n)\}$ is linearly independent, as we wanted to show. $\square$

Of course you may just show both the facts that β spans W and that β is linearly independent. (The two facts are proven along the way in the two approaches above.) That is correct, too.

3. (16 pts) Consider the ordered bases $\beta = \{5 - 3x, 3 - 2x\}$ and $\gamma = \{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$, and the linear map $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ given by $T(f(x)) = (f(1), f(1) - 2f(0))$.

(1) Compute the matrix $[T]_\beta^\gamma$.

We have $[T]_\beta^\gamma = ([T(5 - 3x)]_\gamma \ [T(3 - 2x)]_\gamma)$.

Plugging in $5 - 3x$ and $3 - 2x$ in place of $f(x)$, we compute

$$T(5 - 3x) = (2, -8), \qquad T(3 - 2x) = (1, -5).$$

Since γ is standard, the γ-coordinates are the standard coordinates. Thus

$$[T]_\beta^\gamma = \boxed{\begin{pmatrix} 2 & 1 \\ -8 & -5 \end{pmatrix}}.$$

(2) Let $\beta' = \{1, x\}$. Compute the change of coordinate matrix, changing β'-coordinates into β-coordinates (for the vector space $P_1(\mathbb{R})$).

We need to write elements of β' as linear combinations of those of β. So we set up equations

$$1 = a(5 - 3x) + b(3 - 2x), \qquad x = c(5 - 3x) + d(3 - 2x).$$

Simplifying the right hand sides,

$$1 = (5a + 3b) - (3a + 2b)x, \qquad x = (5c + 3d) - (3c + 2d)x.$$

This yields the system of linear equations

$$5a + 3b = 1, \quad 3a + 2b = 0, \quad 5c + 3d = 0, \quad 3c + 2d = -1.$$

Solving this, we obtain $a = 2$, $b = -3$, $c = 3$, $d = -5$.

Answer: $\boxed{\begin{pmatrix} 2 & 3 \\ -3 & -5 \end{pmatrix}}$.

(Another way: One may begin with computing the change of coordinate matrix from β to β', which is seen to be $\begin{pmatrix} 5 & 3 \\ -3 & -2 \end{pmatrix}$ (without computation!). Then you may compute the inverse matrix to arrive at the same answer – that this is true was seen in the lecture; see the sentence above Example 1, p.112.)

4. Let $A \in M_{m\times n}(F)$. (Hint: Consider the left multiplication transformation.)

(1) (6 pts) Show that $W = \{x \in F^n : Ax = 0\}$ is a subspace of F^n. (Here an element x of F^n is viewed as a length n column vector so that the matrix multiplication Ax makes sense.)

The null space is a subspace of the domain (Theorem 2.1) and the map L_A is linear (Theorem 2.15). Hence $W = N(L_A)$ is a subspace of F^n. $\square$

(Of course you can also check that W contains 0, and is closed under addition and scalar multiplication.)

(2) (10 pts) Assume that $n > m$. Prove that $\dim W \geq n - m$.

By the dimension theorem,

$$\dim W + \dim R(L_A) = n.$$

Since $R(L_A)$ is a subspace of F^m,

$$\dim R(L_A) \leq \dim(F^m) = m.$$

Therefore

$$\dim W = n - \dim R(L_A) \geq n - m. \qquad \square$$

5. (16 pts) Let $V = \mathbb{R}^3$, and define $f_1, f_2, f_3 \in V^*$ as follows:

$$f_1(x, y, z) = x - z, \qquad f_2(x, y, z) = 2x + y, \qquad f_3(x, y, z) = -3x + z.$$

(1) Prove that $\beta^* = \{f_1, f_2, f_3\}$ is a basis for V^*.

Since $\dim V^* = \dim V = 3$, it is enough to show $\{f_1, f_2, f_3\}$ is linearly independent for showing $\{f_1, f_2, f_3\}$ is a basis for V^*. Suppose

$$af_1 + bf_2 + cf_3 = 0.$$

Then

$$(af_1 + bf_2 + cf_3)(x, y, z) = (a + 2b - 3c)\,x + by + (-a + c)\,z = 0, \quad x, y, z \in \mathbb{R}.$$

So $a + 2b - 3c = 0$, $b = 0$, and $-a + c = 0$. Solving this system of linear equations, we get $a = b = c = 0$. Thus $\{f_1, f_2, f_3\}$ is linear independent, and a basis. $\square$

(2) Let $\beta = \{v_1, v_2, v_3\}$ be the ordered basis for $\mathbb{R}^3$ such that β^* is the dual basis of β. Compute v_3.

Let $v_3 = (a, b, c)$. By the definition of the dual basis, $f_i(v_j) = \delta_{ij}$. In particular (when $j = 3$)

$$f_1(a, b, c) = 0, \quad f_2(a, b, c) = 0, \quad f_3(a, b, c) = 1.$$

so

$$a - c = 0, \quad 2a + b = 0, \quad -3a + c = 1.$$

Solving the equation, we get $a = c = -1/2$, $b = 1$. Answer: $\boxed{v_3 = \left(-\frac{1}{2}, 1, -\frac{1}{2}\right)}$.