Math 110 (Fall 2018) Midterm I (Monday September 24, 12:10-1:00)

Name: SID:

Please write clearly and legibly. Justify your answers. Partial credits may be given to Problems 2, 3, 4, and 5. When submitting the exam, include scratch paper if you put solutions there or if you think you may get partial credit from scratch work. In that case write your name on scratch paper to avoid a mix-up.

Rules during the exam:

- Do NOT start until everyone receives the exam and the proctors give the green light.
- It is NOT allowed to leave the room between 12:53–1:00, to minimize distraction for others.
- If you have a question, raise your hand and a proctor will come to you.
- If it turns out during the exam that a problem needs extra clarification, it will be put on the blackboard for everyone to see.

Tips for your solutions:

- You're free to use theorems in chapters 1 and 2 of the textbook without proofs.
- When using the facts from exercises, you should justify them.
- Using tools from outside chapters 1 and 2 is not encouraged, but if you utilize them: you'll get full credit if that leads to a complete solution, but little partial credit for incomplete work.
- It helps to explain what you're doing. Phrases like "It is enough to prove ... because ..." "I'm going to prove by contradiction. So let's suppose that ..." (or "Toward a contradiction, suppose that ..."), "Let's prove the 'if' part. ... Now we prove the 'only if' part." and so on are all useful. Clearly indicate logical relations between sentences if applicable.
- Certain shorthand notation may be used to save writing. E.g. $A \Rightarrow B$ means "A implies B", and $A \Leftrightarrow B$ means "A if and only if B" (that is, A implies B, and B implies A), $\forall$ means "for all (for every)", $\therefore$ for "therefore" and so on. If you're not sure your usage of symbol is standard, it's always a good idea to write out what you mean.

Below some common notation is recalled.

- m, n are always positive integers,
- F is a field (every vector space or linear map is considered over F unless specified otherwise),
- F^n is the vector space of all n-tuples (or length n column vectors) with entries in F,
- $\{e_1, ..., e_n\}$ is the standard ordered basis for F^n,
- $M_{m\times n}(F)$ is the vector space of all $m \times n$ matrices with entries in F,
- A^t is the transpose of a matrix A,
- L_A is the left multiplication transformation of a matrix A,
- $\mathcal{L}(V, W)$ is the vector space of all linear maps from V to W,
- $P_n(F)$ is the vector space of polynomials with coefficients in F of degree at most n,
- V^* is the dual vector space of V,
- T^t is the transpose of a linear map T.

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"As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others."

1. Mark each of the following (1)–(4) True (T) or False (F). Don't give a full proof but provide a brief justification with main points, no more than a few sentences. (Correct with justification = 4 pts, Correct but no or wrong justification = 2 pts, Incorrect answer = 0 pt.) See page 1 for notation.

(1) (   ) Let V be a finite-dimensional vector space, β a basis for V, and W a subspace of V. Then there is a subset of β which is a basis for W.

(2) (   ) Let $T : V \to W$ and $U : W \to Z$ be linear maps of finite dimensional vector spaces. If UT is invertible, then either U or T (or possibly both) is invertible.

(3) (   ) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ (over $\mathbb{R}$) such that $R(T) = N(T)$.

(4) (   ) Let $T : V \to W$ be a linear map of finite dimensional vector spaces. If $T^t : W^* \to V^*$ is the zero transformation then T is also the zero transformation.

2. (16 pts) Let $T : V \to W$ be an isomorphism, where V, W are finite dimensional vector spaces. If β is a basis for V, prove that $T(\beta) = \{T(x) : x \in \beta\}$ is a basis for W.

3. (16 pts) Consider the ordered bases $\beta = \{5 - 3x, 3 - 2x\}$ and $\gamma = \{(1, 0), (0, 1)\}$ for $\mathbb{R}^2$, and the linear map $T : P_1(\mathbb{R}) \to \mathbb{R}^2$ given by $T(f(x)) = (f(1), f(1) - 2f(0))$.

(1) Compute the matrix $[T]_\beta^\gamma$.

(2) Let $\beta' = \{1, x\}$. Compute the change of coordinate matrix, changing β'-coordinates into β-coordinates (for the vector space $P_1(\mathbb{R})$).

4. Let $A \in M_{m\times n}(F)$. (Hint: Consider the left multiplication transformation.)

(1) (6 pts) Show that $W = \{x \in F^n : Ax = 0\}$ is a subspace of F^n. (Here an element x of F^n is viewed as a length n column vector so that the matrix multiplication Ax makes sense.)

(2) (10 pts) Assume that $n > m$. Prove that $\dim W \geq n - m$.

5. (16 pts) Let $V = \mathbb{R}^3$, and define $f_1, f_2, f_3 \in V^*$ as follows:

$$f_1(x, y, z) = x - z, \qquad f_2(x, y, z) = 2x + y, \qquad f_3(x, y, z) = -3x + z.$$

(1) Prove that $\beta^* = \{f_1, f_2, f_3\}$ is a basis for V^*.

(2) Let $\beta = \{v_1, v_2, v_3\}$ be the ordered basis for $\mathbb{R}^3$ such that β^* is the dual basis of β. Compute v_3.