MATH 185 LECTURE 4 FINAL EXAM SOLUTIONS

FALL 2017

Name: ______________________________

Exam policies:

- Please write your name on each page.
- Closed book, closed notes, no external resources, individual work.
- Be sure to justify any yes/no answers with computations and/or by appealing to the relevant theorems. One word answers will *not* receive full credit.
- The usual expectations and policies concerning academic integrity apply.
- You may use any theorem proved in class unless the problem states otherwise.
- Since there are several slightly different conventions for the "Cayley transform", write down the map explicitly whenever you need it in conformal mapping problems.

(1) (20 points, 5 each) Prove or disprove each of the following statements.
   (a) If γ is a smooth closed curve in $\mathbb{C} \setminus \{0\}$, then

$$\int_\gamma \frac{1}{z^4}\, dz = 0$$

   (b) If $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions which converges to $f : \Omega \to \mathbb{C}$ uniformly on each compact subset $K \subset \mathbb{C}$, then f must be holomorphic.
   (c) The half plane $\{z : \text{Re}(z) > \text{Im}(z)\}$ is conformally equivalent to the half-strip $\{z : \text{Re}(z) < 0,\ 0 < \text{Im}(z) < 1\}$. Either construct a suitable conformal map (do not appeal to the Riemann mapping theorem) or prove that no such map exists.
   (d) The half plane $\{z : \text{Re}(z) > \text{Im}(z)\}$ is conformally equivalent to $\mathbb{C}$. (Same remark as before.)

Solution. (a) *True. The function* $f(z) = \frac{1}{z^4}$ *has a primitive* $F(z) = -\frac{1}{3z^3}$ *in* $\mathbb{C} \setminus \{0\}$*, so the integral of* f *over any closed curve is zero by the fundamental theorem of calculus.*

(b) *True. It suffices to prove that* f *is holomorphic on each disc* D *whose closure* $\overline{D}$ *is contained in* Ω*. This follows from Morera's theorem: if* $T \subset D$ *is an oriented triangle, then by Cauchy-Goursat and the fact that* $f_n \to f$ *uniformly on* T*,*

$$\int_T f\, dz = \lim_{n\to\infty} \int_T f_n\, dz = 0.$$

(c) *True. We can map the half strip* $\Omega = \{z : \text{Re}(z) < 0, |\text{Im}(z)| < 1\}$ *to the half plane* $\{z : \text{Re}(z) > \text{Im}(z)\}$ *as follows. First apply* $z \mapsto \exp(\pi z)$ *to obtain the upper half disc, then apply the FLT* $z \mapsto \frac{1+z}{1-z}$ *to map the upper half disc to the first quadrant, then apply* $z \mapsto e^{-3\pi i/4} z^2$ *to obtain the half plane* $\{\text{Re}(z) > \text{Im}(z)\}$*.*
[TODO: picture]

(d) *False. If* $f : \mathbb{C} \setminus \{z : \text{Re}(z) > \text{Im}(z)\}$ *is a conformal equivalence, then* $i \notin \overline{f(\mathbb{C})}$*, so* $g(z) = \frac{1}{f(z)-i}$ *is bounded and entire. By Liouville's theorem,* g *is constant and nonzero, so* $f(z) - i$ *is constant.*

(2) (10 points) Consider a function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(\theta) = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + a_3 e^{3i\theta}$, where $\theta \in \mathbb{R}$ and $a_j \in \mathbb{C}$ with $a_3 \neq 0$. Prove that there exists $\theta \in \mathbb{R}$ such that $|f(\theta)| > |a_0|$. [Hint: relate f to a suitable function of a complex variable.]

Solution. *The problem is equivalent to showing that if $F(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$ for $z \in \mathbb{C}$, then there exists z^* with $|z^*| = 1$ such that $|F(z^*)| > |a_0|$, as the required θ is then obtained by writing $z^* = e^{i\theta}$ in polar form.*

But since $|F(0)| = |a_0|$ and F is not constant (as $a_3 \neq 0$), the maximum modulus principle implies that $\sup_{|z|\leq 1} |F(z)| = \sup_{|z|=1} |F(z)| > |a_0|$.

(3) (10 points)

(a) (5 points) Determine the radius of convergence of the power series

$$\sum_{k=1}^{\infty} k^2 \cos\Big(\frac{k\pi}{2}\Big) z^k.$$

(b) (5 points) Determine the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!}(z+1)^n, \text{ where } f(z) = \frac{z^3+8}{(z+2)(z^2+4)}.$$

Solution. (a) *We apply Hadamard's formula: if R is the radius of convergence of the power series $\sum_n a_n z^n$, then $R^{-1} = \limsup_{n\to\infty} |a_n|^{1/n}$.. Since*

$$\limsup_{k\to\infty} \Big| k^2 \cos\Big(\frac{k\pi}{2}\Big) \Big|^{1/k} = \limsup_{k\to\infty} k^{2/k} = \limsup_{k\to\infty} e^{\frac{2\log k}{k}} = 1,$$

where in the second inequality we use the fact that $|\cos(\frac{k\pi}{2})| = 1$ for all k even, it follows that the radius of convergence is 1.

(b) *Writing*

$$f(z) = \frac{(z+2)(z-2e^{i\pi/3})(z-2e^{5\pi i/3})}{(z+2)(z+2i)(z-2i)},$$

we see that f has an analytic continuation to $z=-2$ given explicitly by

$$F(z) = \frac{(z-2e^{i\pi/3})(z-2e^{5\pi i/3})}{(z+2i)(z-2i)},$$

which is analytic on every disc $D_r(1)$ which does not contain $\pm 2i$. On the other hand, $\lim_{z\to\pm 2i} |F(z)| = \infty$ so F has no analytic continuation to any $D_r(-1)$ which contains $\pm 2i$. Therefore the radius of convergence of the power series, which is also the power series for $F(z)$, is $|\pm 2i + 1| = \sqrt{5}$.

(4) (10 points) Determine the poles, their orders, and the residues of the function

$$f(z) = \frac{z}{e^z - 1}.$$

[Hint: Taylor expansion may help with some computations.]

Solution. *The function $g(z) = e^z - 1$ has zeros at $z = 2\pi k$ for each integer k. As $g'(2\pi ik) = e^{2\pi ik} = 1$ is never zero, these zeros are all simple, so for each k we can write $g(z) = (z - 2\pi k) + (z - 2\pi k)^2 r_k(z)$ where $r_k(z)$ is holomorphic and nonvanishing at $2\pi ik$.*

Each of the zeros of g is an isolated singularity of f. Since in a neighborhood of 0 we have $f(z) = \frac{z}{z + zr_0(z)} = \frac{1}{1 + r_0(z)}$, the singularity at 0 is removable.

In a neighborhood of $2\pi ik$ for nonzero k we can write

$$f(z) = \frac{z}{(z - 2\pi ik)[1 + (z - 2\pi ik)r_k(z)]},$$

So f has a simple pole at $2\pi ik$, and

$$\text{Res}(f, 2\pi ik) = \lim_{z \to 2\pi ik} (z - 2\pi ik) f(z) = \lim_{z \to 2\pi ik} \frac{z}{1 + (z - 2\pi ik) r_k(z)} = 2\pi ik.$$

(5) (10 points) Suppose f is entire and satisfies the bound $|f(2^{-k})| \leq 2^{-k^2}$ for all positive integers k.

(a) (2 points) Show that $f(0) = 0$.

(b) (5 points) Show that in fact $f^{(n)}(0) = 0$ for all $n = 1, 2, 3 \dots$.

(c) (3 points) Prove that $f(z) = 0$ for all z.

Solution. (a) *By continuity,* $f(0) = \lim_{k\to\infty} f(2^{-k}) = 0$.

(b) *Suppose* $f^{(n)}(0)$ *are not all zero. Let* $N > 0$ *be the smallest integer such that* $f^{(N)}(0) \neq 0$*; thus* 0 *is a zero of order* N*, and we can write* $f(z) = z^N g(z)$ *where* g *is holomorphic and nonvanishing in a neighborhood of* 0*. Then for all* k *large enough we have*

$$2^{-k^2} \geq |f(2^{-k})| = 2^{-Nk}|g(2^{-k})| \geq 2^{-Nk}c$$

for some positive $c > 0$*, so*

$$c \leq 2^{Nk-k^2}.$$

But since $\lim_{k\to\infty} Nk - k^2 = -\infty$*, the right side of the inequality goes to* 0 *as* $k \to \infty$*, which yields a contradiction.*

(c) *Since* f *is entire, its Taylor series expansion at any point converges to* $f(z)$ *for all* z*. So* $f(z) = \sum_n \frac{f^{(n)}(0)}{n!} z^n = 0$.

(6) (10 points) Find a conformal map from the open region between the two circles $|z - i| = 1$ and $|z - 4i| = 4$ to the unit disc. You may leave your answer as a composition of explicit "elementary" maps as we have done in class. Please label all relevant geometric quantities (e.g. x or y intercepts of lines, center and radii of circles) in any sketches.

Solution. *First apply the inversion $z \mapsto \frac{1}{z}$ to map the point of tangency to ∞. Since*

$$|z - i|^2 = 1 \Leftrightarrow |z|^2 - i\bar{z} + iz + 1 = 1 \Rightarrow 1 - i\Big(\frac{1}{z} - \frac{1}{\bar{z}}\Big) = 0,$$

and

$$|z - 4i|^2 = 4 \Leftrightarrow |z|^2 - 4i\bar{z} + 4iz + 16 = 16 \Rightarrow 1 - 4i\Big(\frac{1}{z} - \frac{1}{\bar{z}}\Big) = 0,$$

the image of the circles $|z-i| = 1$ and $|z-4i| = 4$ under the map $w = \frac{1}{z}$ satisfy the equations

$$1 + 2\,\mathrm{Im}(w) = 0, \quad 1 + 8\,\mathrm{Im}(w) = 0,$$

respectively. Thus the region between the circles is mapped to the strip

$$\Big\{-\frac{1}{2} < \mathrm{Im}(w) < -\frac{1}{8}\Big\}.$$

The map $z \mapsto \frac{8\pi}{3}(z + \frac{1}{2}i)$ takes this strip to the horizontal strip

$$\{0 < \mathrm{Im}(z) < \pi\},$$

whereupon applying $\exp(z)$ followed by the Cayley transform $z \mapsto \frac{z-i}{z+i}$ maps this to the unit disc.

Summing up, the composition $f_4 \circ f_3 \circ f_2 \circ f_1$, where

$$f_1(z) = \frac{1}{z},\ f_2(z) = \frac{8\pi i}{3}\Big(z + \frac{1}{2}\Big),\ f_3(z) = \exp(z),\ f_4(z) = \frac{z - i}{z + i},$$

maps the region between the two circles to the unit disc.

(7) (10 points) Determine the number of zeros of the function $p(z) = z^7 - 4z^2 + 15z - 8i$ in the annulus $\{1 < |z| < 2\}$.

Solution. *On the circle $|z| = 2$, we compare p to the function $q_1(z) = z^7$:*

$$|p(z) - q_1(z)| \le |-4z^2 + 15z - 8i| \le 4(2)^2 + 15(2) + 8 = 54 < |z|^7 = 128.$$

By Rouche's theorem, p has 7 zeros in the disc $|z| < 2$.

On $|z| = 1$, compare p instead to the function $q_2(z) = 15z$:

$$|p(z) - q_2(z)| \le |z^7 - 4z^2 - 8i| \le 1 + 4 + 8 < 15 = |15z|.$$

Rouche implies that p has 1 zero in $|z| < 1$, and the above inequality implies that p does not vanish on $|z| = 1$.

Consequently p has 6 zeros in the region $\{1 < |z| < 2\}$.

(8) (10 points) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{3+\cos(\theta)}.$$

Solution. *Make the subtitution* $\cos(\theta) = \frac{1}{2}(z+z^{-1})$, $z = e^{i\theta}$, $\frac{dz}{iz} = d\theta$, *to write*

$$\int_0^{2\pi} \frac{d\theta}{3+\cos(\theta)} = \int_{|z|=1} \frac{1}{3+\frac{z+z^{-1}}{2}} \cdot \frac{1}{iz}\,dz = \frac{2}{i}\int_{|z|=1} \frac{1}{z^2+6z+1}\,dz.$$

The integrand has simple poles at $z_\pm = \frac{-6\pm\sqrt{32}}{2} = -3\pm\sqrt{8}$; *only* $z_+ = -\frac{1}{3+\sqrt{8}}$ *lies inside the unit disc, and*

$$\mathrm{Res}\Big(\frac{1}{z^2+6z+1}, z_+\Big) = \lim_{z\to z_+} \frac{1}{z-z_-} = \frac{1}{z_+ - z_-} = \frac{1}{2\sqrt{8}}.$$

Consequently, the integral equals

$$\frac{2}{i}\cdot 2\pi i\,\mathrm{Res}\Big(\frac{1}{z^2+6z+1}, z_+\Big) = \frac{\pi}{\sqrt{2}}.$$

(9) (10 points) Let $t > 0$ be a positive number. Using the residue theorem, evaluate

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} \frac{e^{st}}{s^2+1}\, ds,$$

where $\gamma_R = [1 - iR, 1 + iR]$ is the line from $1 - iR$ to $1 + iR$. [Hint: to decide how to introduce an additional curve obtain a closed contour, pay attention to where the integrand is small and where it is large, keeping in mind that $t > 0$, so that you can estimate the integral over that curve in the limit as $R \to \infty$.]

Solution. *Let $\Gamma_R = \gamma_R + C_R$ be the boundary of the left half disc of radius R centered at 1; thus C_R is the semicircular arc defined by $|s - 1| = R$, $\mathrm{Re}(s) \leq 1$.*

On one hand, we evaluate

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{e^{st}}{s^2+1}\, ds$$

by the residue theorem. The integrand $f(s) = \frac{e^{st}}{(s-i)(s+i)}$ has simple poles at $s = \pm i$, and

$$\mathrm{Res}(f, i) = \lim_{s\to i} \frac{e^{st}}{s+i} = \frac{e^{it}}{2i}, \quad \mathrm{Res}(f, -i) = \lim_{s\to -i} \frac{e^{st}}{s-i} = -\frac{e^{-it}}{i}.$$

Therefore

$$\frac{1}{2\pi i} \int_{\Gamma_R} f(s)\, ds = \frac{e^{it} - e^{-it}}{2i} = \sin(t).$$

On the other hand,

$$\int_{\Gamma_R} f(s)\, ds = \int_{\gamma_R} f(s)\, ds + \int_{C_R} f(s)\, ds,$$

and since $|e^{st}| = |e^{t\,\mathrm{Re}(s)} e^{it\,\mathrm{Im}(s)}| \leq e^t$ is bounded on C_R uniformly in R,

$$\left|\int_{C_R} f(s)\, ds\right| \leq \pi R \sup_{s\in C_R} \left|\frac{e^{st}}{s^2+1}\right| \leq \pi R e^t \sup_{s\in C_R} \frac{1}{\left||s|^2 - 1\right|} \leq \frac{\pi R e^t}{(R-1)^2 - 1},$$

where the last inequality follows from the observation that $|s - 1| = R$ implies $|s| \geq R - 1$. Consequently the integral over C_R goes to 0 as $R \to \infty$, and we conclude that

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{\gamma_R} f(s)\, ds = \lim_{R\to\infty} \int_{\Gamma_R} f(s)\, ds = \sin(t).$$

Extra page for work

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