MATH 185 LECTURE 4 MIDTERM 2

FALL 2017

Name: ____________________________________

Exam policies:

- Please write your name on each page.
- Closed book, closed notes, no external resources, individual work.
- Be sure to justify any yes/no answers with computations and/or by appealing to the relevant theorems. One word answers will *not* receive full credit.
- The usual expectations and policies concerning academic integrity apply.
- All problems are weighted equally.

(1) Let

$$f(z) = \frac{1}{z^4 + 8z^2 + 16}.$$

Determine the poles of f, their orders, and the residue of f at each pole.

Solution. Factoring

$$f(z) = \frac{1}{(zr + 4)^2} = \frac{1}{(z - 2i)^2 (z + 2i)^2},$$

we see that f has double poles (i.e. order 2) at $\pm 2i$.

$$\mathrm{Res}(f, 2i) = \lim_{z \to 2i} \frac{d}{dz}[(z - 2i)^2 f(z)] = \lim_{z \to 2i} \left[-\frac{2}{(z + 2i)^3} \right] = \frac{1}{32i}$$

$$\mathrm{Res}(f, -2i) = \lim_{z \to -2i} \frac{d}{dz}[(z + 2i)^2 f(z)] = \lim_{z \to -2i} \left[-\frac{2}{(z - 2i)^3} \right] = -\frac{1}{32i}.$$

(2) Evaluate the integral

$$\int_0^\pi \frac{1}{4-\sin(2x)}\,dx.$$

Solution. Writing $\sin(2x) = \frac{e^{2ix}-e^{-2ix}}{2i}$, and noting that $\zeta = e^{2ix},\ x \in [0,\pi]$ parametrizes the unit circle, we have

$$\int_0^\pi \frac{1}{4-\sin(2x)}\,dx = \int_{|\zeta|=1} \frac{1}{4-\frac{\zeta-\zeta^{-1}}{2i}}\frac{d\zeta}{2i\zeta} = -\int_{|\zeta|=1}\frac{d\zeta}{\zeta^2-8i\zeta-1}.$$

The roots of the polynomial $\zeta^2 - 2i\zeta - 1$ are

$$\zeta = \frac{8i \pm \sqrt{-64+4}}{2} = (4 \pm \sqrt{15})i,$$

and only $(4-\sqrt{15})i$ is enclosed by the circle $|\zeta| = 1$. Therefore, applying the residue theorem (or the Cauchy integral formula) yields

$$-\int_{|\zeta|=1}\frac{d\zeta}{(\zeta-(4+\sqrt{15})i)(\zeta-(4-\sqrt{15})i)} = -2\pi i \cdot \frac{1}{(4-\sqrt{15})i-(4+\sqrt{15})i} = \frac{\pi}{\sqrt{15}}.$$

(3) Suppose $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and satisfies the bound $|f(z)| \geq |z| - 10$ for all $z \in \mathbb{C}$. Prove that for each $w \in \mathbb{C}$, there exists a sequence $\{z_n\}_n \subset \mathbb{C}$ such that $\lim_{n\to\infty} f(z_n) = w$.

Solution. Suppose this were not the case, so that there exist $w \in \mathbb{C}$ and $\delta > 0$ such that $\inf_{z\in\mathbb{C}} |f(z) - w| \geq \delta$. Then $g(z) := \frac{1}{f(z)-w}$ is entire, and $|g(z)| \leq \delta^{-1}$ for all z. Thus by Liouville's theorem, $g(z) \equiv g(0)$ for all $z \in \mathbb{C}$. Since $g(0) = \frac{1}{f(0)-w} \neq 0$, it follows that $f(z) = \frac{1}{g(0)} + w$ is constant, which contradicts the hypothesis that $\lim_{z\to\infty} |f(z)| = \infty$.

(4) Let $\Omega \subset \mathbb{C}$ be open and $\Gamma \subset \Omega$ be a cycle homologous to zero in Ω. Prove that if $f : \Omega \to \mathbb{C}$ is holomorphic, then

$$W_\Gamma(z) f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \text{ for all } z \in \Omega \setminus \Gamma.$$

Solution. Fix $z \in \Omega \setminus \Gamma$. Let $\overline{D}_\varepsilon(z) \subset \Omega \setminus \Gamma$ be a small closed disc centered at z which does not intersect Γ. Then the integrand $\zeta \mapsto \frac{f(\zeta)}{\zeta - z}$ is holomorphic on $\Omega \setminus \{z\}$, and the cycle $\Gamma - W_\Gamma(z)\partial D_\varepsilon$ is nullhomologous in $\Omega \setminus \{z\}$. To see this last claim, note that $W_\Gamma(\zeta) = 0 = W_{\partial D_\varepsilon(z)}(\zeta)$ if $\zeta \in \mathbb{C} \setminus \{z\}$, while $W_{\Gamma - W_\Gamma(z)\partial D_\varepsilon}(z) = W_\Gamma(z) - W_\Gamma(z) W_{\partial D_\varepsilon(z)}(z) = 0$. Therefore, by the general Cauchy integral theorem and the Cauchy integral formula for discs,

$$\begin{aligned} 0 &= \int_{\Gamma - W_\Gamma(z)\partial D_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta - W_\Gamma(z) \int_{\partial D_\varepsilon(z)} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \int_\Gamma \frac{f(\zeta)}{\zeta - z} - 2\pi i W_\Gamma(z) f(z). \end{aligned}$$

(5) Define $f : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ by $f(z) = \sqrt[3]{z} := \exp(\frac{1}{3}\operatorname{Log}(z))$. Let

$$P_1(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(8+8i)}{n!}(z-8-8i)^n, \quad P_2(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(-8-8i)}{n!}(z+8+8i)^n$$

Determine the radius of convergence of P_1 and P_2, and compute $P_1(-8) + P_2(-8)$.

Solution. Since f is holomorphic on the disc $D_{|8+8i|}(8+8i)$, the radius of convergence of P_1 is at least $|8+8i| = 8\sqrt{2}$. On the other hand, f has no analytic extension to any disc centered at $8+8i$ which contains 0, for $\lim_{z\to 0} |f'(z)| = \lim_{z\to 0} \frac{1}{3}|z|^{-2/3} = \infty$. Thus the radius of convergence of P_1 is exactly $8\sqrt{2}$.

For P_2, argue similarly with $f_2(z) := \exp(\frac{1}{3}\log_{(-2\pi,0)}(z)),\ z \in \mathbb{C} \setminus [0, \infty)$, which agrees with f in a neighborhood of $-8-8i$ and is holomorphic on the disc $D_{8\sqrt{2}}(-8-8i)$, but has no analytic extension to any larger disc. The radius of convergence is therefore $8\sqrt{2}$ as well.