MATH 185 LECTURE 4 MIDTERM 1

FALL 2017

Name: ________________________________

Exam policies:

- Please write your name on each page.
- Closed book, closed notes, no external resources, individual work.
- Be sure to justify any yes/no answers with computations and/or by appealing to the relevant theorems. One word answers will *not* receive full credit.
- The usual expectations and policies concerning academic integrity apply.

(1) Evaluate the following integrals:

(a)

$$\int_\gamma \frac{z^2}{z^2+2z-8}\,dz,$$

where γ is the (counterclockwise) radius 3 circle centered at $z=i$.

Solution. Factoring the denominator of the integrand, we can write

$$\int_{|z+i|=3} \frac{z^2}{z^2+2z-8}\,dz = \int_{|z+i|=3} \frac{f(z)}{(z-2)}\,dz,$$

where $f(z) = \frac{z^2}{z+4}$ is holomorphic on a disc containing the circle $|z+i| = 3$, which encloses the point $z = -2$. The integral therefore equals $2\pi i f(2) = \frac{4\pi i}{3}$ by Cauchy's integral formula.

(b)

$$\int_\gamma \frac{\overline{z}}{z+2i}\,dz,$$

where γ is the radius 1 circle (oriented counterclockwise) centered at $z=-2i$.

Solution. Note that the integrand is not holomorphic so the integration theorems we have learned do not apply. Therefore we just proceed from the definition of the contour integral. We parametrize the contour γ by $z(t) = -2i + e^{it},\ t \in [0, 2\pi]$, and compute

$$\int_\gamma \frac{\overline{z}}{z+2i}\,dz = \int_0^{2\pi} 2i + e^{-it}\,idt = -4\pi.$$

(2) State Hadamard's formula for the radius of convergence of the power series $\sum_n a_n z^n$. Then compute the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{4^n z^{2n}}{n^2+n}.$$

Solution. Hadamard's formula says that the radius R of convergence is

$$R = \frac{1}{\limsup_n |a_n|^{1/n}}.$$

For the above series one has $a_{2m} = \frac{2^{2m}}{m^2+m}$ and $a_{2m+1} = 0$ for all integers m. Using the bound $n^2 \leq n^2+n \leq 2n^2$ and the limit $\lim_{n\to\infty}(Cn^2)^{1/n} = \lim_{n\to\infty} e^{\frac{\log n}{n}} = 1$ for any constant $C > 0$, we obtain

$$\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{2n\to\infty}\left(\frac{2^{2n}}{n^2+n}\right)^{\frac{1}{2n}} = \lim_{n\to\infty} \frac{2}{(n^2+n)^{1/2n}} = 2.$$

Therefore the radius of convergence is $\frac{1}{2}$.

(3) Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \sqrt{|\operatorname{Re}(z)||\operatorname{Im}(z)|}$. Prove that f is not holomorphic at the origin $z = 0$.

Solution. By the definition of the derivative, we need to show that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

does not exist. Take h of the form $h = t + it$ for real t. Then

$$\frac{f(h) - f(0)}{h} = \frac{|t|}{(1+i)t} = \begin{cases} \frac{1}{1+i}, & t > 0 \\ -\frac{1}{1+i}, & t < 0 \end{cases}$$

Therefore the limit as $t \to 0$ does not exist.

(4) Compute the complex derivative f' of the following functions:

(a) $f(z) = \exp(\sin(z^3))$ for $z \in \mathbb{C}$.
Solution: By the chain rule, $f'(z) = \exp(\sin(z^3))\cos(z^3) \cdot 3z^2$.

(b) $f(z) = \log(z^2 - 1)$, for $z \in \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ (take it as given that $\log(z^2 - 1)$ is well-defined on this domain).
Solution. Recalling that $\log'(z) = \frac{1}{z}$ for any branch of log, we apply the chain rule again to obtain

$$f'(z) = \frac{2z}{z^2 - 1}.$$

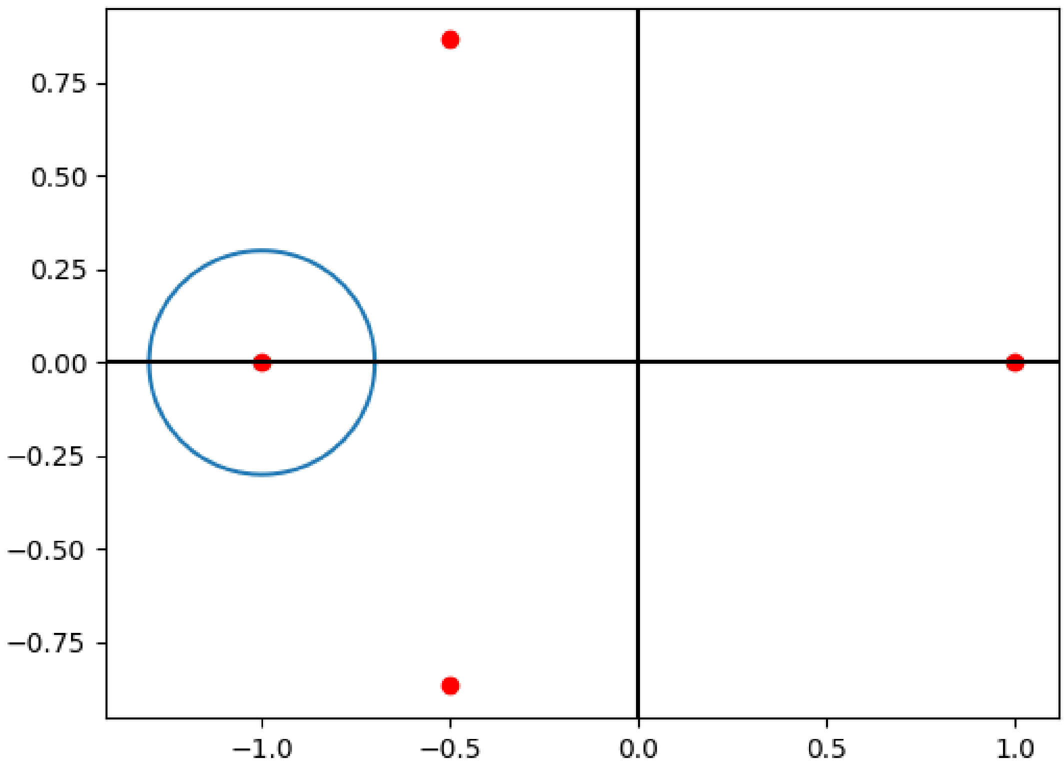

FIGURE 1.

(5) Let K be the set of zeros of the function $f(z) = (z^3 - 1)(z^2 - 1)$. Find K, and then prove that there does not exist a branch of $\log((z^3 - 1)(z^2 - 1))$ for $z \in \mathbb{C} \setminus K$.

Solution. $f(z) = 0$ iff either $z^3 - 1 = 0$ or $z^2 - 1 = 0$. The solutions to the first equation are $z = 1, e^{2\pi i/3}, e^{4\pi i/3}$, while the second equation has solutions $z = \pm 1$. Overall we find

$$K = \{\pm 1, e^{2\pi i/3}, e^{4\pi i/3}\}.$$

Suppose for contradiction that $g(z) = \log((z^3 - 1)(z^2 - 1))$ is well-defined. on $\mathbb{C} \setminus K$. By the chain rule, we have

$$g'(z) = \frac{f'(z)}{f(z)} = \frac{3z^2(z^2 - 1) + 2z(z^3 - 1)}{(z^3 - 1)(z^2 - 1)} = \frac{3z^2}{z^3 - 1} + \frac{2z}{z^2 - 1}.$$

Let C is a small counterclockise circle centered at $z = -1$ and enclosing no other points of K, as shown in the figure above. On one hand, the Fundamental Theorem of Calculus implies that

$$\int_C g'(z)\, dz = 0.$$

On the other hand, we have

$$\int_C g'(z)\, dz = \int_C \frac{3z^2}{z^3 - 1}\, dz + \int_C \frac{2z}{z^2 - 1}.$$

As C does not enclose any zeros of $z^3 - 1$ the function $z \mapsto \frac{3z^2}{z^3-1}$ is holomorphic on a small disc containing C. Therefore the first integral on the right is zero by Cauchy's integral theorem. For the second integral, write $\frac{2z}{z^2-1} = \frac{2z}{(z-1)(z+1)}$ and apply Cauchy's integral formula to obtain

$$\int_C \frac{2z}{z^2 - 1} = 2\pi i \frac{2(-1)}{(-1) - 1} = 2\pi i.$$

Overall this computation shows that $\int_C g'(z)\, dz = 2\pi i$, which is a contradiction.