MATH 126 FINAL EXAM

Name: ______________________________

Exam policies:

- Closed book, closed notes, no external resources, individual work.
- Please write your name on the exam and on each page you detach.
- Unless stated otherwise, you must justify all answers with computations or by appealing to the relevant theorems.
- You may use any theorem presented in class unless the problem states otherwise.
- The usual expectations and policies concerning academic integrity apply.
- Notation: $\Box = \partial_t^2 - \Delta_x$.
- Recall *Euler's identity*: $e^{i\theta} = \cos\theta + i\sin\theta$

(1) For $n = 1, 2, \ldots$, let $\phi_n(x) := \sqrt{2}\sin(n\pi x)$. Show that $\{\phi_n\}_n$ are orthonormal with respect to the inner product

$$\langle f, g \rangle := \int_0^1 f(x)\overline{g(x)}\, dx.$$

($\overline{z}$ denotes complex conjugate).

Solution. Recall that ϕ_n are eigenfunctions of $-\partial_{xx}$: $-\partial_{xx}[\sin(n\pi x)] = n^2\pi^2 \sin(n\pi x)$. Thus when $n \neq m$, we integrate by parts to find

$$\begin{aligned} -n^2\pi^2\langle \phi_n, \phi_m\rangle = \langle -\partial_{xx}\phi_n, \phi_m\rangle = \langle \partial_x\phi_n, \partial_x\phi_m\rangle = \langle \phi_n, -\partial_{xx}\phi_m\rangle = -m^2\pi^2\langle \phi_n, \phi_m\rangle, \\ \Rightarrow (n-m)\langle \phi_n, \phi_m\rangle = 0 \\ \Rightarrow \langle \phi_n, \phi_m\rangle = 0. \end{aligned}$$

(2) Solve the Schrödinger equation on $(0, 1)$ with Dirichlet boundary conditions:

$$\begin{cases} iu_t + u_{xx} = 0, \ (t, x) \in (0, \infty) \times (0, 1), \\ u(0, x) = 4\sin(\pi x) + \sin(2\pi x), \\ u(t, 0) = u(t, 1) = 0, \end{cases}$$

where $i^2 = -1$ (your solution will be complex valued). Also compute

$$\int_0^1 |u(t, x)|^2 \, dx.$$

Solution. We first construct solutions in the separated form

$$u_1 = p_1(t)\sin(\pi x), \quad u_2 = p_2(t)\sin(2\pi x),$$

and combine them using linearity. The p_j are determined by substituting u_j into the equation and using the fact that $\sin(n\pi x)$ are eigenfunctions of $-\partial_{xx}$. The resulting ODEs

$$ip_1' - \pi^2 p_1 = 0, \quad p_1(0) = 1$$
$$ip_2' - 4\pi^2 p_2 = 0, \quad p_2(0) = 1$$

yield

$$p_1(t) = e^{-i\pi^2 t}, \ p_2(t) = e^{-i4\pi^2 t},$$

so $u_1 = e^{-i\pi^2 t}\sin(\pi x), \ u_2 = e^{-i4\pi^2 t}\sin(2\pi x)$, and by linearity the solution to the original problem is

$$u = 4u_1 + u_2 = 4e^{-i\pi^2 t}\sin(\pi x) + e^{-i4\pi^2 t}\sin(2\pi x).$$

Finally, by Problem 1 and the Pythagorean theorem

$$\int_0^1 |u(t, x)|^2 \, dx = \langle u, u \rangle = 8 + \frac{1}{2} = \frac{17}{2}.$$

(3) Let $m : \mathbb{R}^3 \to \mathbb{C}$ be a smooth function and write $m(D)$ for the corresponding Fourier multiplier operator. Show that if there exists $M > 0$ such that $|m(\xi)| \leq M$ for all $\xi \in \mathbb{R}^3$, then

$$\Big(\int_{\mathbb{R}^3} |m(D)u|^2 \, dx\Big)^{1/2} \leq M\Big(\int_{\mathbb{R}^3} |u|^2 \, dx\Big)^{1/2} \text{ for any } u \in \mathcal{S}(\mathbb{R}^3).$$

Solution. By Plancherel,

$$\int |m(D)u|^2 \, dx = \int |m(\xi)\widehat{u}(\xi)|^2 \, d\xi \leq M^2 \int |\widehat{u}(\xi)|^2 \, d\xi = M^2 \int |u|^2 \, dx.$$

(4) Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

(a) Show that if u *strict subsolution* to Laplace's equation in the sense that $-\Delta u < 0$ in Ω, then

(*) $$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

You should clearly state, but need not prove, any facts from calculus that you use.

(b) Show that (*) holds if u merely satisfies $-\Delta u \leq 0$.

Note: do *not* just quote the maximum principle; the point here is to prove it.

Solution. If $z_0 \in \mathbb{R}^2$ is a local maximum of a C^2 function v, then $\Delta v(z_0) \leq 0$.

First we prove the conclusion under the assumption that $-\Delta u < 0$. Assume for contradiction that there exists $z_0 \in \Omega$ such that $u(z_0) > \max_{\partial\Omega}$. Then the $\max_{\overline{\Omega}} u$ is attained at some interior point $z_* \in \Omega$. But then $\Delta u(z_0) \leq 0$, contrary to the hypothesis.

Assume now that $-\Delta u \leq 0$. By hypothesis $\Omega \subset \{|x| \leq R\}$ for some R, for each $\varepsilon > 0$ let $u_\varepsilon = u + \varepsilon^{x_1}$. As $\Delta u_\varepsilon = \Delta u + \varepsilon e^{x_1} \geq \Delta u + e^{-R} > 0$, the first part implies that the conclusion holds for each u_ε, and since $\max_\Omega |u_\varepsilon - u| \leq \varepsilon e^R \to 0$ as $\varepsilon \to 0$, the equality holds for u as well.

(5) The Laplacian of a function u in spherical coordinates (r, θ, ϕ) is

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2 \frac{\partial}{\partial r} u\Big) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\Big(\sin\theta \frac{\partial}{\partial \theta} u\Big) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} u.$$

Our coordinate convention here is $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$. Let $R > 1$, V_1, V_2 be constants, and let Ω be the region between the concentric spheres $r = 1$ and $r = R$. Solve the boundary value problem

$$\begin{cases} \Delta u = 0, \text{ in } \Omega, \\ u|_{r=1} = V_1, u|_{r=R} = V_2. \end{cases}$$

In a sentence or two, explain briefly why the solution you find is the *only* solution belonging to $C^2(\Omega) \cap C(\overline{\Omega})$.

Solution. Look for solutions of the form $u = R(r)$. Then

$$0 = \frac{\partial}{\partial r}\Big(r^2 \frac{\partial}{\partial r} u\Big),$$

so

$$R' = \frac{A}{r^2}, \quad R(r) = B - \frac{A}{r}.$$

Applying the boundary conditions

$$V_1 = B - A, \quad V_2 = B - \frac{A}{R}$$

so

$$A = \frac{V_2 - V_1}{1 - \frac{1}{R}}, \ B = V_1 + A = V_1 + \frac{V_2 - V_1}{1 - \frac{1}{R}} = \frac{V_2 - \frac{V_1}{R}}{1 - \frac{1}{R}}$$

so overall

$$u = R(r) = \frac{V_2 - \frac{V_1}{R}}{1 - \frac{1}{R}} + \Big(\frac{V_1 - V_2}{1 - \frac{1}{R}}\Big)\frac{1}{r}.$$

Uniqueness follows from the uniqueness theorem for Laplace's equation

(6) Consider the conservation law

$$\begin{cases} u_t + 2uu_x = 0, & (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = g(x), \end{cases} \qquad g(x) = \begin{cases} 1, & x < 0 \\ -2, & x \geq 0. \end{cases}$$

Compute the characteristics starting from the x axis, and find a weak solution.

Solution. The characteristics satisfy

$$\begin{cases} \dot{t} = 1, & t(0) = 0 \\ \dot{x} = 2z, & x(0) = x_0, \\ \dot{z} = 0, & z(0) = g(x_0). \end{cases}$$

Thus

$$(t,x,z) = (s, x_0 + 2sg(x_0), g(x_0)).$$

We construct a piecewise constant solution of the form

$$u(t,x) = \begin{cases} 1, & x < \xi(t) \\ -2, & x > \xi(t) \end{cases}$$

for some smooth curve $x = \xi(t)$ with $\xi(0) = 0$. Since u would be a classical solution on either side of the curve, for u to be a weak solution on $(0,\infty) \times \mathbb{R}$ we need only apply the Rankine-Hugoniot condition to determine ξ.

Write the equation as $u_t + (u^2)_x u = 0$. The Rankine-Hugoniot condition then requires

$$1^2 - (-2)^2 = \dot{\xi}(t)(1 - (-2)),$$

so $\dot{\xi} = -1$, hence $\xi(t) = -t$.

(7) Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Evaluate $\lim_{n\to\infty} nf(nx)$ in the sense of distributions.

Solution. Let ϕ be a test function.

$$\begin{aligned} \int nf(nx)\phi(x)\,dx &= \frac{1}{n}\int_{-\frac{1}{n}}^{\frac{1}{n}} \phi(0)\,dx + \frac{1}{n}\int_{-\frac{1}{n}}^{\frac{1}{n}} [\phi(x) - \phi(0)]\,dx \\ &= 2\phi(0) + r_n, \end{aligned}$$

where, by the fundamental theorem of calculus

$$|r_n| \leq \max|\phi'| \int_{-\frac{1}{n}}^{\frac{1}{n}} \frac{|x|}{n}\,dx \leq \frac{C}{n} \to 0 \text{ as } n \to \infty.$$

Consequently

$$\lim_{n\to\infty} \int nf(nx)\phi(x)\,dx = 2\phi(0);$$

in other words, $nf(nx)$ converges in distribution to $2\delta_0$.

(8) Suppose u solves $\Box u = 0$ for $(t,x) \in (0,\infty) \times \mathbb{R}^2$, and that

$$\int_{|x|>2} |\partial_t u(0,x)|^2 + |\nabla_x u(0,x)|^2 \, dx = 0, \qquad \lim_{|x|\to\infty} u(t,x) = 0.$$

Describe the largest region in $(0,\infty) \times \mathbb{R}^2$ on which u is guaranteed to be zero. You may use formulas and/or a sketch that clearly labels all relevant geometric quantities.

Solution. Since $|\nabla u(0,x)| = 0$ for $|x| > 2$, by integrating along radial lines from infinity and using the second hypothesis we deduce that $u(0,x) = 0$, $\partial_t u(0,x) = 0$ when $|x| > 2$. By finite speed of propagation, (considering backwards light cones with base in the region $|x| > 2$) we conclude that $u \equiv 0$ in the region

$$\{(t,x) \in (0,\infty) \times \mathbb{R}^2 : |x| > 2 + t.\}$$

Extra space for work