MATH 126 MIDTERM EXAM 2

Name: ________________________________

Exam policies:

- Closed book, closed notes, no external resources, individual work.
- Please write your name on the exam and on each page you detach.
- Unless stated otherwise, you must justify all answers with computations or by appealing to the relevant theorems.
- You may use any theorem presented in class and the homeworks unless the problem states otherwise.
- The usual expectations and policies concerning academic integrity apply.
- Notation: $\Box = \partial_t^2 - \Delta_x$

(1) Let u and v be smooth functions such that

$$(-\partial_t^2 + \partial_x^2)u = 0 \text{ on } (0,\infty)_t \times \mathbb{R}_x, \quad (-\partial_t^2 + \partial_x^2)v = 0 \text{ on } (0,\infty)_t \times \mathbb{R}_x.$$

Suppose we know that $u(0,x) = v(0,x)$ and $\partial_t u(0,x) = \partial_t v(0,x)$ for all $|x| \geq 1$. Sketch the largest region in $\{(t,x) \in [0,\infty) \times \mathbb{R},\ t \geq 0\}$ on which u and v are guaranteed to agree. Be sure to describe all relevant geometric quantities (e.g. equations of lines) and justify your answers.

Solution. *The function* $w := u - v$ *satisfies* $\Box w = 0$ *and* $w(0,x) = \partial_t w(0,x) = 0$ *for* $|x| \geq 1$. *By the d'Alembert formula*

$$w(t,x) = \frac{1}{2}\big(w(0,x-t) + w(0,x+t)\big) + \frac{1}{2}\int_{x-t}^{x+t} \partial_t w(0,y)\, dy,$$

$w = 0$ *whenever* $(x-t, x+t) \cap (-1,1)$ *is empty, i.e. whenever* $1 \leq x - t$ *or* $x + t \leq -1$.

(2) Evaluate $\lim_{t\to\infty} u(t,x)$, where u is the solution to

$$\begin{cases} u_t - u_{xx} = 0 \text{ in } (0,\infty)\times(0,1), \\ u(0,x) = 1 - 2\cos(3\pi x), \\ u_x(t,0) = u_x(t,1) = 0 \text{ for } t \geq 0. \end{cases}$$

Solution. *Since*

$$u(t,x) = 1 - 2e^{-9\pi^2 t}\cos(3\pi x),$$

we see that $\lim_{t\to\infty} u(t,x) = 1$ *for all* x.

(3) Let $\Omega \subset \mathbb{R}^2$ be bounded and open with boundary $\partial\Omega$. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$-(1+y^2)(\partial_x^2 + \partial_y^2)u - x^2\partial_x u < 0 \text{ on } \Omega.$$

Prove that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u.$$

You should clearly state, but need not prove, any theorems from calculus that you use.

Solution. *Suppose the inequality failed. Then there exists some interior local maximum* $(x_0, y_0) \in \Omega$ *such that* $u(x_0, y_0) > \max_{\partial\Omega} u$*. At this local maximum we have* $\partial_x u(x_0, y_0) = \partial_y u(x_0, y_0) = 0$ *and, by the second derivative test,* $\partial_x^2 u(x_0, y_0) \leq 0$ *and* $\partial_y^2 u(x_0, y_0) \leq 0$*. This yields the contradiction*

$$-(1+y^2)(\partial_x^2 + \partial_y^2)u(x_0, y_0) - x_0^2\partial_x u(x_0, y_0) \geq 0.$$

(4) Assume $g : \mathbb{R}^3 \to R$ is bounded and continuous, and let $u \in C^\infty\big((0,1)\times\mathbb{R}^3\big)$ be a bounded solution to the nonlinear heat equation

$$\begin{cases} (\partial_t - \Delta)u = u^2 - u^3 & \text{in } (0,1)\times\mathbb{R}^3, \\ u(0,x) = g(x) & \text{on } \{t=0\}\times\mathbb{R}^3. \end{cases}$$

Find $\varepsilon > 0$ such that if $|g(x)| \le \varepsilon$ for all x, then $|u(t,x)| \le 2\varepsilon$ for all $(t,x) \in (0,1)\times\mathbb{R}^3$.

Solution. *We show that any $\varepsilon \le \frac{1}{8}$ works.*

By hypothesis, there exists $M > 0$ such that $|u(t,x)| \le M$ for all $(t,x) \in [0,1]\times\mathbb{R}^3$. Let T be the largest time in $[0,1]$ such that $|u(t,x)| \le 2\varepsilon$ for all $(t,x) \in [0,T]\times\mathbb{R}^3$. By the Duhamel formula

$$u(t,x) = \int_{\mathbb{R}^3} \Phi(t, x-y) g(y)\, dy + \int_0^t \int_{\mathbb{R}^3} \Phi(t-s, x-y)[u(s,y)^2 - u(s,y)^3]\, dydx$$

and the bounds $\left|\int_{\mathbb{R}^3} \Phi(t,x-y)g(y)\,dy\right| \le \varepsilon \int_{\mathbb{R}^3} \Phi(t,x-y)\,dy \le \varepsilon$, and similarly for the nonlinear term, we obtain

$$|u(t,x)| \le \varepsilon + t(M^2 + M^3),$$

and see that $T \ge \frac{\varepsilon}{M^2+M^3} > 0$.

Assume for contradiction that $T < 1$. By the Duhamel formula,

$$|u(T,x)| \le \varepsilon + (2\varepsilon)^2 + (2\varepsilon)^3 \le \varepsilon + 4\varepsilon^2 + 8\varepsilon^3 \le (1 + \tfrac{1}{2} + \tfrac{1}{8})\varepsilon = (2 - \tfrac{3}{8})\varepsilon.$$

But we can then apply the Duhamel formula to the equation initialized at time T,

$$u(t,x) = \int_{\mathbb{R}^3} \Phi(t-T, x-y) u(T,y)\, dy + \int_T^t \int_{\mathbb{R}^3} \Phi(t-T, x-y))[u(s,y)^2 - u(s,y)^3]\, dyds, \ t \ge T,$$

(note that $v(t,x) := u(t+T,x)$ solves $(\partial_t - \Delta)v = v^2 - v^3$ on $(0, 1-T)\times\mathbb{R}^3$ with initial data $v(0,x) = u(T,x)$) to deduce that

$$|u(t,x)| \le (2 - \tfrac{3}{8})\varepsilon + (t-T)(M^2 + M^3) < 2\varepsilon$$

whenever $t - T < \frac{3\varepsilon}{8(M^2+M^3)}$, which contradicts the maximality of T.

(5) Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with smooth boundary $\partial\Omega$. Suppose u is a smooth function on the cylinder $[0,T] \times \overline{\Omega}$ for some $T > 0$ such that $u = 0$ on $[0,T] \times \partial\Omega$. Derive the energy estimate

$$E(t) \leq E(0) + \int_0^t \int_\Omega |\partial_t u(t,x)||\Box u(t,x)|\, dxds \text{ for all } 0 \leq t \leq T,$$

where

$$E(t) = \frac{1}{2}\int_\Omega |\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2\, dx.$$

Solution.

$$\begin{aligned} E'(t) &= \int_\Omega \partial_t u \partial_t^2 u + \langle \nabla u, \nabla \partial_t u \rangle\, dx \\ &= \int_\Omega \partial_t u(\partial_t^2 - \Delta) u\, dx, \end{aligned}$$

so by the Fundamental Theorem of Calculus

$$\begin{aligned} E(t) &\leq E(0) + \Big| \int_0^t E'(s)\, ds \Big| \leq E(0) + \int_0^t |E'(s)|\, ds \\ &\leq E(0) + \int_0^t \int_\Omega |\partial_t u||\Box u|\, dxds \end{aligned}$$