MATH 126 MIDTERM EXAM 1

Name: ______________________________

Exam policies:

- Closed book, closed notes, no external resources, individual work.
- Please write your name on the exam and on each page you detach.
- Unless stated otherwise, you must justify all answers with computations or by appealing to the relevant theorems.
- You may use any theorem presented in class unless the problem states otherwise.
- The usual expectations and policies concerning academic integrity apply.

(1) Let $H = \chi_{[0,\infty)}$ be the indicator function of $[0,\infty)$; in other words

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Let G be the distribution $\mathcal{D}(\mathbb{R}^2) \ni \phi \mapsto \int_{\mathbb{R}^2} H(x-t)\phi(t,x)\,dxdt$. Evaluate $(\partial_t + \partial_x)G$ and $(\partial_x - \partial_t)G$.

Solution. For a test function $\phi \in \mathcal{D}(\mathbb{R}^2)$,

$$(\partial_t + \partial_x)G(\phi) = -\int_{\mathbb{R}^2} H(x-t)(\partial_t + \partial_x)\phi\,dxdt.$$

We have

$$\int_{\mathbb{R}^2} H(x-t)\partial_t\phi\,dxdt = \int_{-\infty}^{\infty}\int_{-\infty}^{x} \partial_t\phi(t,x)\,dtdx = \int_{-\infty}^{\infty} \phi(x,x)\,dx,$$
$$\int_{\mathbb{R}^2} H(x-t)\partial_x\phi dxdt = \int_{-\infty}^{\infty}\int_{t}^{\infty} \partial_x\phi\,dxdt = -\int_{-\infty}^{\infty} \phi(t,t)\,dt.$$

Therefore $(\partial_t + \partial_x)G = 0$, while

$$(\partial_x - \partial_t)G(\phi) = 2\int_{-\infty}^{\infty} \phi(x,x)\,dx.$$

(2) Let $u = \frac{1}{2}\chi_{[-1,1]}$, where $\chi_{[-1,1]}$ is the indicator function for the interval $[-1,1]$, and $v(x) = |x|$. Evaluate the convolution $u * v(x)$ for $x \in \mathbb{R}$.

Solution. When $x \in [-1,1]$,

$$\begin{aligned} u * v(x) = \frac{1}{2}\int_{-1}^{1} |x-y|\, dy &= \frac{1}{2}\Big(\int_{-1}^{x} x - y\, dy + \int_{x}^{1} y - x\, dy\Big) \\ &= \frac{1}{2}\Big(x(x+1) - \Big(\frac{x^2-1}{2}\Big) + \frac{1-x^2}{2} - x(1-x)\Big) \\ &= \frac{x^2+1}{2}. \end{aligned}$$

When $x \notin [-1,1]$,

$$\begin{aligned} x > 1 \Rightarrow u * v(x) &= \frac{1}{2}\int_{-1}^{1} x - y\, dy = x, \\ x < 1 \Rightarrow u * v(x) &= \frac{1}{2}\int_{-1}^{1} y - x\, dy = -x. \end{aligned}$$

Summing up,

$$u * v(x) = \begin{cases} |x|, & |x| > 1 \\ \frac{x^2+1}{2}, & |x| < 1 \end{cases}$$

(3) Let $u \in C^2(\mathbb{R}^2)$ be a solution to $\Delta u = 0$ on $\mathbb{R}^2$ such that u is constant on all curves C_r of the form

$$C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 = r^2\},$$

for all $r > 0$. Prove that u is constant on $\mathbb{R}^2$.

Solution. By the maximum principle applied to u and $-u$, u is constant on each solid ellipse $\Omega_r = \{(x, y) : x^2 + 2y^2 < r^2\}$; indeed

$$\min_{C_r} \leq \min_{\Omega_r} u \leq \max_{\Omega_r} u \leq \max_{C_r} u,$$

and the leftmost and rightmost expressions are equal. Any (x, y) is contained in some Ω_r, hence $u(x, y) = u(0, 0)$ for all (x, y).

(4) Let $f_n(x) = n^2 \cos(nx)$. Evaluate the limit $\lim_{n\to\infty} f_n$ in the sense of distributions.

Solution. If ϕ is a test function, then repeatedly integrating by parts (on a large interval containing the support of ϕ) we compute

$$\begin{aligned}\int_{\mathbb{R}} n^2 \cos(nx)\phi(x)\,dx &= -\int_{\mathbb{R}} n \sin(nx)\phi'(x)\,dx = -\int_{\mathbb{R}} \cos(nx)\phi''(x)\,dx \\ &= \frac{1}{n}\int_{\mathbb{R}} \sin(nx)\phi^{(3)}(x)\,dx \to 0 \text{ as } n \to \infty.\end{aligned}$$

(5) Let $g \in C^1(\mathbb{R})$. Using the method of characteristics, solve the initial value problem

$$\begin{cases} (1+t)u_t + u_x = t, \\ u(0,x) = g(x), \end{cases} \qquad (t,x) \in (0,\infty) \times \mathbb{R}.$$

Check that the function you derive in terms of g is indeed a solution.

Solution. The characteristics defined by the ODE

$$\dot{t} = 1 + t, \quad \dot{x} = 1,$$

and initialized to $\big(t(0), x(0)\big) = (0, x_0)$ are $t = e^s - 1,\ x = x_0 + s$, so $s = \log(1+t),\ x_0 = x - \log(1+t)$. Setting $z(s) = u\big(t(s), x(s)\big)$, the PDE implies have $\dot{z}(s) = t(s) = e^s - 1,\ z(0) = g(x_0)$, so

$$u(t,x) = z = e^s - s - 1 + g(x_0) = t - \log(1+t) + g\big(x - \log(1+t)\big).$$

Indeed

$$u_t = 1 - \frac{1}{1+t} - \frac{g'\big(x - \log(1+t)\big)}{1+t}, \quad u_x = g'\big(x - \log(1+t)\big),$$

and so

$$(1+t)u_t + u_x = t, \quad u(0,x) = g(x).$$