MATH 1A MIDTERM 2 (001) 12.10PM - 1PM
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS FINISHED

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

Name and section: ____

GSI's name: ____

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Calculate the following:

(a) (10 points)

$$\frac{d}{dx}(\arcsin(\sqrt{1-x^2}))$$

Solution:

$$\frac{d}{dx} \arcsin\left(\sqrt{1-x^2}\right)$$

$$= \frac{1}{\sqrt{1-(1-x^2)}} \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)$$

(b) (15 points)

$$\lim_{x\to\infty}(x - \ln(2x))$$

Solution:

$$\lim_{x\to\infty}(x - \ln(2x)) = \lim_{x\to\infty} x\left(1 - \frac{\ln(2x)}{x}\right)$$

$$\lim_{x\to\infty} \frac{\ln(2x)}{x} = \lim_{x\to\infty} \frac{\left(\frac{2}{2x}\right)}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} x = \infty \quad \underline{\text{and}} \quad \lim_{x\to\infty}\left(1 - \frac{\ln(2x)}{x}\right) = 1$$

$$\Rightarrow \lim_{x\to\infty} x - \ln(2x) = \infty$$

2. (25 points) Find the equation of the tangent line to the following curve at the given point.

$$\sin(x+2y) = x - 2y, \qquad (\pi, \pi/2).$$

Solution:

$$\sin(x+2y) = x - 2y \Rightarrow \frac{d}{dx}\left(\sin(x+2y)\right) = \frac{d}{dx}(x-2y)$$

$$\Rightarrow \left(1 + 2\frac{dy}{dx}\right)\cos(x+2y) = 1 - 2\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \cos(x+2y)}{2 + 2\cos(x+2y)}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=\pi \\ y=\pi/2}} = \frac{1 - \cos(2\pi)}{2 + 2\cos(2\pi)} = 0$$

$\Rightarrow$ Equation of tangent is $y = \pi/2$.

3. (25 points) Sketch of the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{x^3}{x^2 - x}$$

Solution:

$$f(x) = \begin{cases} \frac{x^2}{x-1} & x \neq 0 \\ \text{DNE} & x = 0 \end{cases}$$

Lets graph $y = \frac{x^2}{x-1}$ and remove $(0,0)$ at end

$g(x) = \frac{x^2}{x-1}$ Domain $= \mathbb{R}$ minus 1

x and y intercept $(0,0)$ (to be removed later)

slant asymptote $m = 1$ $b = \lim_{x \to \infty, -\infty} \left(\frac{x^2}{x-1} - x\right)$

$= \lim_{x \to \infty, -\infty} \frac{x^2 - x(x-1)}{(x-1)} = \lim_{x \to \infty, -\infty} \frac{x}{x-1} = 1 \Rightarrow y = x+1$ slant

$\lim_{x \to 1^+} \frac{x^2}{x-1} = \infty$, $\lim_{x \to 1^-} \frac{x^2}{x-1} = -\infty$

$$g'(x) = \frac{2x(x-1) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2} = \frac{x(x-2)}{(x-1)^2}$$

$+ \quad 0 \quad - \quad 1 \quad - \quad 2 \quad +$ $g'(x)$ $\Rightarrow$ $f(0)$ local max, $f(2)$ local min

$$g''(x) = \frac{(2x-2)(x-1)^2 - 2(x-1)(x^2-2x)}{(x-1)^4}$$

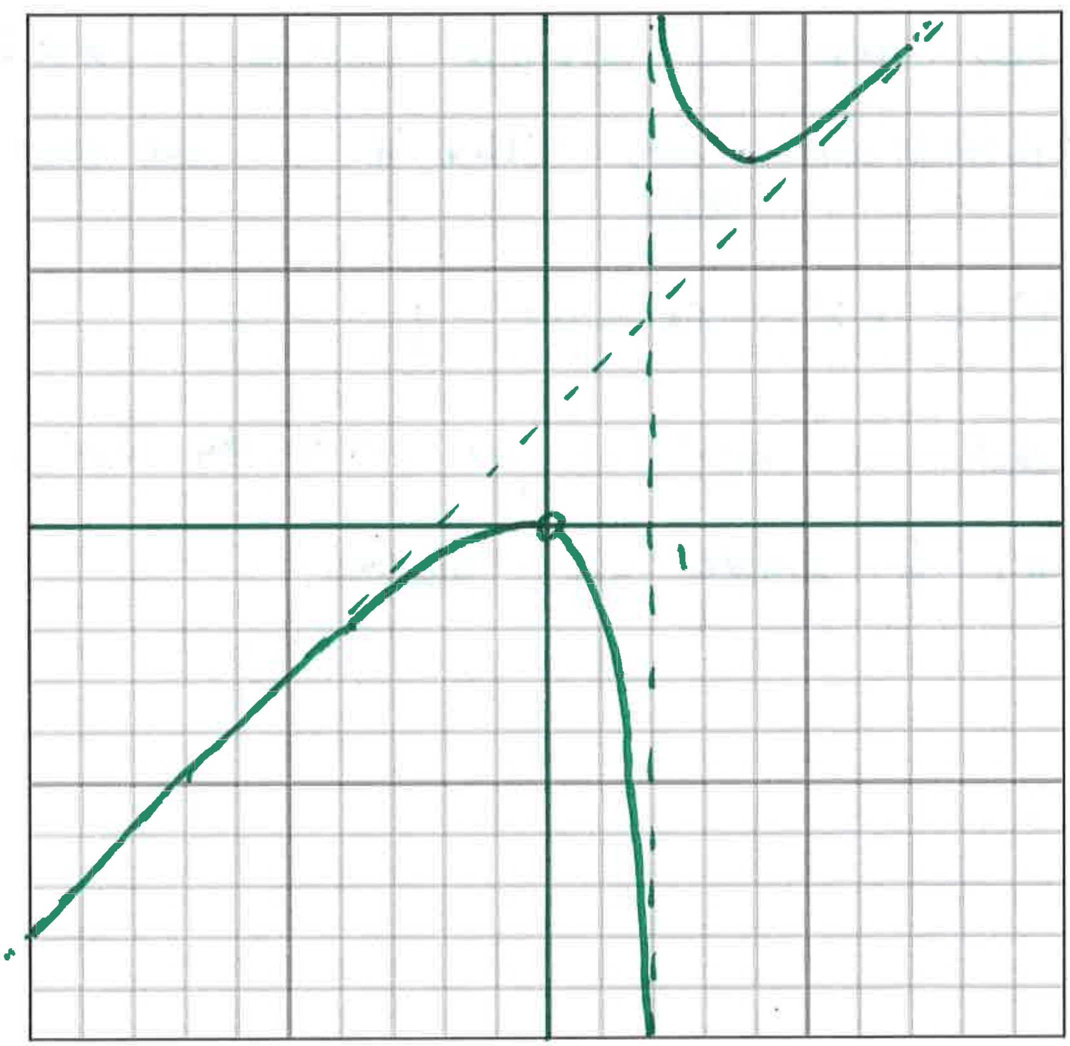
Solution (continued) :

$$= \frac{(2x-2)(x-1) - 2(x^2-2x)}{(x-1)^3} = \frac{2x^2 - 2x - 2x + 2 - 2x^2 + 4x}{(x-1)^3}$$

$$= \frac{2}{(x-1)^3}$$

$-$ $+$

1 not inflection



4. (25 points) Show that the following equation has exactly one real solution. Be sure to carefully justify you answer clearly stating any results you use from lectures.

$$\cos^2(x) = 1 + 4x$$

Solution:

$f(x) = \cos^2(x) - 4x - 1$ $\quad (-1 \leq \cos^2(x) \leq 1)$

$f(10) < 0$, $f(-10) > 0$ $\underset{\text{I.V.T}}{\Longrightarrow}$ There exists at least one solution to $f(x) = 0$

Assume there are 2 distinct solutions $a < b$ with $f(a) = f(b) = 0$ $\underset{\text{Rolle}}{\Longrightarrow}$ There is c such that $f'(c) = 0$

But $f'(c) = -2\sin(x)\cos(x) - 4 < 0$

$\Rightarrow$ No such a and b can exist by Rolle.

Hence there is exactly one solution.

5. (25 points) Find the point on the curve $4x^2 + y^2 - 4 = 0$ which is farthest from (0, 1). Hint: Went maximizing the objective function make sure you think about the domain.

Solution:

Q = distance = $\sqrt{x^2 + (y-1)^2}$

Constraint : $4x^2 + y^2 - 4 = 0 \Rightarrow x^2 = \dfrac{4 - y^2}{4} = 1 - \dfrac{y^2}{4}$

$$\Rightarrow Q = \sqrt{1 - \frac{y^2}{4} + (y-1)^2} = f(y) \quad -2 \leq y \leq 2$$

$$\Rightarrow \frac{df}{dy} = \frac{-\frac{y}{2} + 2(y-1)}{2\sqrt{1 - \frac{y^2}{4} + (y-1)^2}} = \frac{\frac{3}{2}y - 2}{2\sqrt{1 - \frac{y^2}{4} + (y-1)^2}}$$

$\frac{3}{2}y - 2 = 0$

$\Rightarrow y = \frac{4}{3}$

$-$ $+$

-2 $\frac{4}{3}$ 2

$\Rightarrow f(\frac{4}{3})$ is a local min.

$\Rightarrow$ Max must be at either $y = -2$ or 2

$f(2) = 1$, $f(-2) = 3$

$y = -2 \Rightarrow x = 0 \Rightarrow$ Farthest point is $(0, -2)$

END OF EXAM

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