Math 53 Final Exam, Prof. Srivastava
May 11, 2018, 11:40pm–2:30pm, 155 Dwinelle Hall.

Name: Nikhil Srivastava

SID: ____________________

GSI: ____________________

Name of the student to your left: ____________________
Name of the student to your right: ____________________

Instructions: Write all answers clearly in the provided space. This exam includes some space for scratch work at the bottom of pages 2 and 6 which will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit. You are allowed to bring one *single-sided handwritten letter size* cheat sheet. Calculators, phones, textbooks, and your own scratch paper are not allowed. **If you are seen writing after time is up, you will lose 20 points.**

When you are done, hand over your exam to your GSI *unless your GSI is Shiyu Li*, in which case hand it over to me.

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Question:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Total
Points:	12	6	6	8	6	6	6	6	6	6	6	6	6	7	7	100

Do not turn over this page until your instructor tells you to do so.

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1. (12 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) Suppose $f(x, y)$ is differentiable and $f_x = 1$ and $f_y = -2$ at a point. Then there is a direction u such that $D_u f = 0$ at that point. **T** **F**

   Choose $= \langle a, b \rangle$ so $a(1) + b(-2) = 0$.

   (b) If the level curve of a differentiable function $g(x, y) = k$ intersects itself non-tangentially at a point P,
   then P must be a critical point of g. **T** **F**

   Since ∇g is $\perp$ to the level curve of g.

   (c) If a and b are vectors in $\mathbb{R}^3$ then $a \times (a \times b)$ is always zero. **T** **F**

   (d) The flux of $F = \langle x, 0, 0 \rangle$ across a sphere of radius 1 at the origin is strictly less than its flux across a sphere of radius 2 at the origin,
   where both are outwardly oriented. **T** **F**

   Because flux $= \iiint_E \operatorname{div}(F)\,dV = 2\operatorname{Vol}(E)$.

   (e) If $\operatorname{curl}(\nabla f) = \nabla f$ for a function f defined on $\mathbb{R}^3$, then f must be a solution of the PDE

   $$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0.$$

   T **F**

   Take div of both sides:
   $0 = \operatorname{div}(\operatorname{curl}(\nabla f)) = \operatorname{div}(\nabla f) =$

   (f) If $S = \{(x, y, z) : f(x, y, z) = k\}$ is a level surface of a smooth function f with no critical points on S, then S must be orientable. **T** **F**

   Take $\vec{n} = \dfrac{\nabla f}{\|\nabla f\|}$, which is a continuous normal vector since $\nabla f \perp S$.

[Scratch Space Below]

2. Determine whether each of the following statements is true. If so, explain why, and if not, provide a counterexample.

   (a) (3 points) If $\mathbf{F}$ and $\mathbf{G}$ are conservative vector fields defined on $\mathbb{R}^3$ then the sum $\mathbf{F}+\mathbf{G}$ is also conservative.

True: There are many ways to see this:

① F and G are conservative, so $\oint_C \bar{F}\cdot d\bar{r}=0$ and $\oint_C \bar{G}\cdot d\bar{r}=0$ for every closed C. Adding these, $\oint_C (\bar{F}+\bar{G})\cdot d\bar{r}=0$ for all such C, so $\bar{F}+\bar{G}$ is conservative.

② By linearity of partial derivatives, $\operatorname{curl}(\bar{F}+\bar{G}) = \operatorname{curl}(\bar{F})+\operatorname{curl}(\bar{G}) = 0+0=\underline{\underline{0}}$.

   (b) (3 points) If $\mathbf{F}=\langle P,Q,R\rangle$ and $\mathbf{G}=\langle S,T,U\rangle$ are conservative vector fields defined on $\mathbb{R}^3$, then the vector field

$$\mathbf{H}=\langle PS,QT,RU\rangle$$

with components equal to their entrywise products, is also conservative.

False: You should be suspicious since entrywise products of vectors rarely had nice properties in this class.

Counterexample: $\bar{F}=\langle y,x,0\rangle$ has curl $=\langle 1-1,0,0\rangle=0$ so it's conservative.

However $\langle y^2,x^2,0\rangle$ has curl $=\langle 2x-2y,0,0\rangle$ which is not zero, so not conservative.

3. (6 points) A particle moves along the intersection of the surfaces:

$$z = x^2 + \frac{y^2}{4} \implies f(x,y,z)=0$$

and

$$x^2 + y^2 = 25. \implies g(x,y,z)=0$$

Suppose its position vector at time t is $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ and we know that $x(0) = 3, y(0) = 4$, and $x'(0) = 4$. Calculate $y'(0)$ and $z'(0)$.

The velocity vector $\bar{r}'(0) = \langle x'(0), y'(0), z'(0)\rangle$ lies in both the planes to $x^2 + \frac{y^2}{4} - z = 0$ and $x^2+y^2-25=0$. The normals to these planes at given by

$$\nabla f = \langle 2x, \tfrac{2y}{4}, -1\rangle = \langle 6, 2, -1\rangle \text{ at } \bar{r}(0),$$

$$\nabla g = \langle 2x, 2y, 0\rangle = \langle 6, 8, 0\rangle \text{ at } \bar{r}(0).$$

Since $\bar{r}'(0)$ must be perpendicular to both of these, it must be parallel to their cross product:

$$\begin{vmatrix} i & j & k \\ 6 & 2 & -1 \\ 6 & 8 & 0 \end{vmatrix} = \langle 8, -6, 36\rangle = c\langle \underset{x'(0),\text{ given}}{4}, y'(0), z'(0)\rangle.$$

Thus we have $C = 1/2$ and

$$y'(0) = -3$$
$$z'(0) = 18 \;//$$

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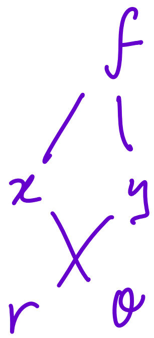
4. Suppose $f(x, y) = xy$ and $x = r\cos\theta, y = r\sin\theta$.

(a) (4 points) Use the chain rule to find the partial derivatives $\partial f/\partial r$ and $\partial f/\partial\theta$.



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$

$$= y\cos\theta + x\sin\theta$$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -yr\sin\theta + xr\cos\theta$$

(b) (4 points) Use this to approximate the value of f at the point $(r, \theta) = (1.001, -0.01)$.

Letting $r_0 = 1$, $\theta_0 = 0$, and $f(r,\theta) := f(r\cos\theta, r\sin\theta)$, we have

$$f(r_0 + \Delta r, \theta_0 + \Delta\theta)$$

$$\approx f(r_0, \theta_0) + \frac{\partial f}{\partial r}\Delta r + \frac{\partial f}{\partial \theta}\Delta\theta$$

At the point $(r,\theta) = (1,0)$ the derivatives are

$\Updownarrow (x,y) = (1,0)$

$f_r = y\cos\theta + x\sin\theta$
$= (0)(1) + (1)(0)$
$= 0$

$f_\theta = -(0) + (1)(1)(1)$
$= 1$

So $f(r = 1.001, \theta = -.01) \approx 1(0) + 0\cdot\underbrace{(.001)}_{\Delta r}$
$+ 1\underbrace{(-.01)}_{\Delta\theta}$

$= -.01$

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5. (6 points) Suppose $z = z(x, y)$ is a differentiable function satisfying $e^z = xyz$. Find $\partial z/\partial x$ and $\partial^2 z/\partial x^2$ as functions of x, y, z.

Treating y as a constant and implicitly differentiating in x:

$$\frac{\partial}{\partial x} e^z = e^z z_x = \frac{\partial}{\partial x}(xyz)$$

$$= yz + yxz_x.$$

Rearranging: $z_x = \dfrac{yz}{e^z - yx}$ //.

Differentiating the above again in x: $\frac{\partial}{\partial x}(e^z z_x) = \frac{\partial}{\partial x}(yz + yxz_x)$

$$\Rightarrow e^z(z_x)^2 + e^z z_{xx} = yz_x + (yxz_{xx} + yz_x)$$

[Scratch Space Below] $\Longrightarrow$ $z_{xx} = \dfrac{2yz_x - e^z(z_x)^2}{e^z - yx}$

$$= \frac{2y\left(yz/(e^z - yx)\right) - e^z\left(\frac{yz}{e^z - yx}\right)^2}{e^z - yx}$$

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6. (6 points) Consider the function $f(x, y) = x^3/3 + y^3/3 + 5x - y$. Find and classify the critical points of the function $g(x, y) = |\nabla f(x, y)|^2$.

$$\nabla f = \langle 3x^2/3 + 5,\ 3y^2/3 - 1 \rangle = \underline{\langle x^2+5,\ y^2-1 \rangle}.$$

So $g(x,y) = x^4 + 10x^2 + 25 + y^4 - 2y^2 + 1$ and

$$\nabla g = \langle 4x^3 + 20x,\ 4y^3 - 4y \rangle$$

So the critical pts are solutions of $4x^3 + 20x = 0 \Rightarrow x(x^2+5x) = 0$
$\Rightarrow x = 0$

$4y^3 - 4y = 0 \Rightarrow y(y^2 - 1) = 0$
$\Rightarrow y = \underline{0, +1, -1}$

So there are <u>three critical pts</u>.

Using $f_{xx} = 12x^2 + 20$, $f_{yy} = 12y^2 - 4$, $f_{xy} = 0$, we have.

	$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$	Type
$(0,0)$	$(20)(-4) < 0$	Saddle pt
$(0,-1)$	$(20)(8) > 0$	local minimum
$(0,1)$	$(20)(8) > 0$	local minimum

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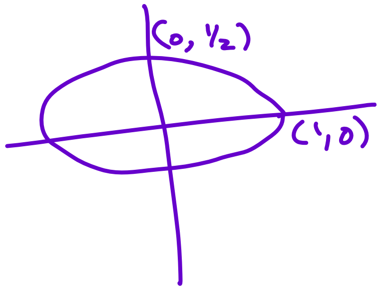
7. (6 points) Find the extreme values of $f(x,y) = e^{-xy}$ in the region $D = \{(x,y) : x^2 + 4y^2 \le 1\}$.



The extreme values are either at critical pts or on the boundary of D.

Since $\nabla f = \langle -ye^{-xy}, -xe^{-xy} \rangle$, the only critical pt is $x=y=0$ where we have $f(0,0) = e^0 = 1$.

To find the optima on the boundary, we use Lagrange multiplies to solve [max/min $f(x,y)$ subject to $g(x,y) = x^2+4y^2 = 1$]:

$$\begin{cases} \langle -ye^{-xy}, -xe^{-xy} \rangle = c\nabla g = c\langle 2x, 8y \rangle & (1) \\ x^2+4y^2 = 1 & (2) \end{cases}$$

(1) $\Longrightarrow$ $\dfrac{-ye^{-xy}}{2x} = \dfrac{-xe^{-xy}}{8y} \Longrightarrow \dfrac{-y}{2x} = \dfrac{-x}{8y}$ since $e^{-xy} \neq 0$

$\Longrightarrow 4y^2 = x^2 \Longrightarrow \underbrace{2x^2 = 1}_{(2)} \Longrightarrow x = \pm\frac{1}{\sqrt{2}}$

$\Longrightarrow y = \pm\frac{1}{\sqrt{8}}$

Checking the values at these pts:

	e^{-xy}
$(1/\sqrt{2}, 1/\sqrt{8})$	$e^{-\frac{1}{4}}$
$(1/\sqrt{2}, -1/\sqrt{8})$	$e^{1/4}$
$(-1/\sqrt{2}, 1/\sqrt{8})$	$e^{1/4}$
$(-1/\sqrt{2}, -1/\sqrt{8})$	$e^{-1/4}$

So the minima are $\pm(1/\sqrt{2}, 1/\sqrt{8})$ with $f = e^{-1/4}$

maxima are $\pm(1/\sqrt{2}, -1/\sqrt{8})$ $f = e^{1/4}$

8. (6 points) Compute the area of the simply connected "moustache" region enclosed by the parameterized curve

$$\mathbf{r}(t) = \langle 5\cos(t), \sin(t) + \cos(4t)\rangle, \quad t \in [0, 2\pi].$$

Let C be the given curve and let D be its interior, whose area we want to compute.

By Green's Theorem: $\displaystyle\oint_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$.

Taking $Q = x$ and $P = 0$, we have

$$\text{Area}(D) = \iint_D 1\,dA = \oint_C x\,dy = \int_0^{2\pi} x(r(t))\,y'(t)\,dt$$

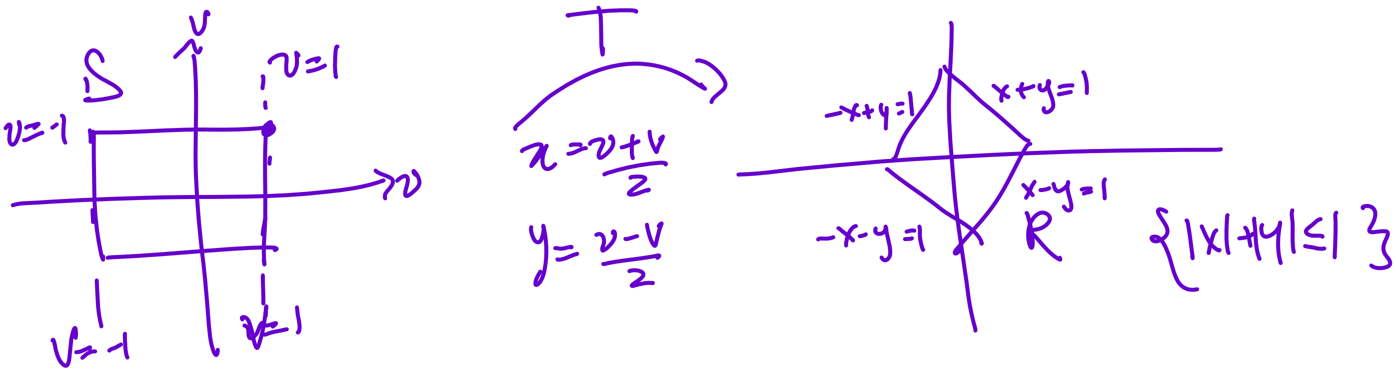
$$= \int_0^{2\pi} 5\cos(t)\big(\cos(t) - 4\sin(4t)\big)\,dt$$

$$= 5\underbrace{\int_0^{2\pi} \cos^2 t\,dt}_{=\frac{\cos(2\theta)+1}{2}\text{ trig identity}} - 20\underbrace{\int_0^{2\pi} \cos(t)\sin(4t)\,dt}_{0 \text{ by integration by parts}} = \boxed{5\pi}.$$

9. (6 points) Evaluate the integral

$$\int\int_R e^{x+y} dA$$

where R is given by the inequality $|x| + |y| \le 1$ by making an appropriate change of variables. (hint: sketch the region first)



The natural change of variables replaces $x+y$ by a single variable:

$$u = x+y, \quad v = x-y \implies x = u+v/2, \quad y = u-v/2.$$

In the uv plane, the lines defining the boundary of R become:

$$x+y=1 \implies u=1 \qquad -x-y=1 \implies u=-1$$
$$x-y=1 \implies v=1 \qquad -x+y=1 \implies v=-1$$

So R is the image of $S = \{-1 \le u \le 1, \ -1 \le v \le 1\}$.

The Jacobian is $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\frac{1}{2}$

So we have $\iint_R e^{x+y} dA = \iint_S e^u \left|-\tfrac{1}{2}\right| du\, dv = \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1} e^u \, dv\, du$

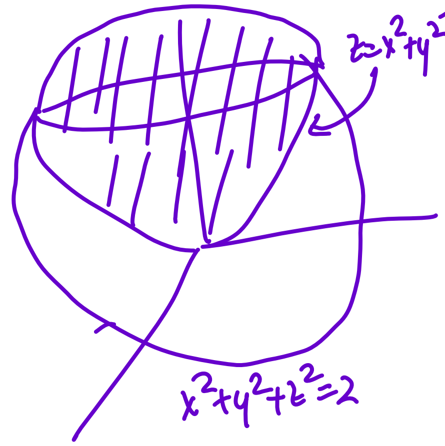
$= \frac{2}{2}\int_{-1}^{1} e^u \, du = \boxed{e - e^{-1}}$

10. (6 points) Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

In cylindrical coords, the surf cas are:

$z = {}^2$

$r^2 + z^2 = 2$



There are two possible regions here, and either was graded as correct. We will use the one <u>above</u> the paraboloid.

So along the curve of intersection we have:

$$r^2 + r^4 = 2 \implies r^2 = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \Rightarrow \underline{\underline{r^2 = 1}}$$

So the shadow of E in the xy-plane is $D = \{ r \leq 1 \}$

Thus the volume is:
$$\iint_D \left(\sqrt{2-r^2} - r^2\right) r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} r\sqrt{2-r^2} - r^3 \, d\theta\, dr = 2\pi \Bigg[\underbrace{\int_0^1 r\sqrt{2-r^2}\,dr}_{\substack{u = 2-r^2 \\ du = -2r\,dr}} - \int_0^1 r^3\,dr \Bigg]$$

$$= 2\pi \left[-\frac{1}{2} \int_2^1 \sqrt{u}\,du - \frac{r^4}{4}\Bigg|_0^1 \right] = 2\pi \left[\frac{1}{2} \frac{u^{3/2}}{3/2}\Bigg|_1^2 - \frac{1}{4} \right] \boxed{= 2\pi \cdot \left(\frac{2\sqrt{2}-1}{3} - \frac{1}{4} \right)}$$

11. (6 points) Find the work done by the force field $\mathbf{F} = \langle z^2, x^2, y^2 \rangle$ on a particle moving along the line segment from $(1, 0, 0)$ to $(4, 1, 2)$.

We parametrize the line segment as:

$$C: \quad \bar{r}(t) = (1-t)\langle 1,0,0\rangle + t\langle 3,1,2\rangle \qquad t\in[0,1]$$

$$= \langle 1+3t, t, 2t\rangle \quad , \quad r'(t) = \langle 3, 1, 2\rangle$$

$$\text{Work} = \int_C \bar{F}\cdot d\bar{r} = \int_0^1 \langle (2t)^2, (1+3t)^2, t^2\rangle \cdot \langle 3,1,2\rangle \, dt$$

$$= \int_0^1 12t^2 + 1 + 9t^2 + 6t + 2t^2 \, dt$$

$$= \int_0^1 23t^2 + 6t + 1 \, dt = 23\frac{t^3}{3} + \frac{6t^2}{2} + t \Big|_0^1$$

$$= \frac{23}{3} + \frac{6}{2} + 1 = \boxed{\frac{35}{3}}$$

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12. (6 points) A surface S is parameterized by

$$\mathbf{r}(u,v) = e^{-u^2}\langle 1, \sin(v), \cos(v)\rangle,$$

where

$$0 \le u \le \sqrt{\pi}, \quad u^2 \le v \le \pi.$$

Find its surface area.

$$\bar{r}_u = -2ue^{-u^2}\langle 1, \sin(v), \cos(v)\rangle$$

$$\bar{r}_v = e^{-u^2}\langle 0, \cos(v), -\sin(v)\rangle$$

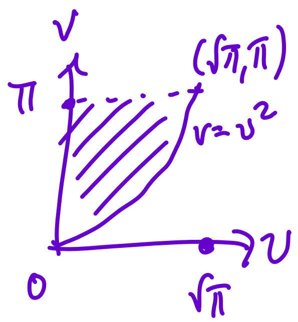
So $\bar{r}_u \times \bar{r}_v = \begin{vmatrix} i & j & k \\ -2ue^{-u^2} & -2ue^{-u^2}\sin(v) & -2ue^{-u^2}\cos(v) \\ 0 & e^{-u^2}\cos(v) & -e^{-u^2}\sin(v) \end{vmatrix}$

$$= \langle 2ue^{-2u^2}\sin^2(v) + 2ue^{-2u^2}\cos^2(v),\ 2ue^{-2u^2}\sin(v),\ -2ue^{-2u^2}\cos(v)\rangle$$

$$= \langle 2ue^{-2u^2},\ 2ue^{-2u^2}\sin(v),\ -2ue^{-2u^2}\cos(v)\rangle \text{ so}$$

$$|\bar{r}_u \times \bar{r}_v| = \left(4u^2e^{-4u^2} + 4u^2e^{-4u^2}\left(\cos^2(v)+\sin^2(v)\right)\right)^{1/2} = \sqrt{8}\,ue^{-2u^2}$$

Thus the area is $\underbrace{\int_0^{\sqrt{\pi}}\int_{u^2}^{\pi} \sqrt{8}\,ue^{-2u^2}\,dv\,du}_{\text{becomes hard to integrate in } v\text{, so switch order}} = \int_0^{\pi}\int_0^{\sqrt{v}} \sqrt{8}\,ue^{-2u^2}\,du\,dv$



$$= \sqrt{8}\int_0^{\pi}\left(\frac{e^{-2u^2}}{-4}\Bigg|_0^{\sqrt{v}}\right)dv$$

$$= \frac{\sqrt{8}}{4}\int_0^{\pi}\left(1 - e^{-2v}\right)dv = \boxed{\frac{2\pi - 1 + e^{-2\pi}}{2\sqrt{2}}}$$

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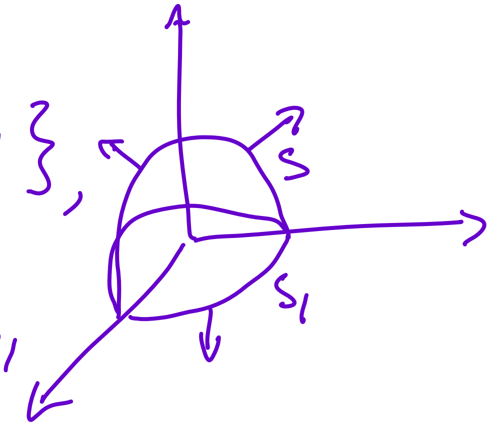
13. (6 points) Find the flux of the vector field

$$\mathbf{F}(x,y,z) = \langle y^3 z, x^3 z, 1 + e^{x^2+y^2} \rangle$$

through the paraboloid part S of the boundary of the solid region

$$E \qquad z + x^2 + y^2 \leq 1; \quad z \geq 0,$$

where S is oriented upwards.



S intersects the xy-plane in: $\{0 + x^2 + y^2 \leq 1\}$, which is the unit disk.

Let S_1 be the unit disk oriented downwards, so that $S \cup S_1$ is the boundary, oriented outwards, of the solid region E.

By the divergence theorem, $\iint_{S_1} \bar{F}\cdot\bar{n}\,dS + \iint_{S} \bar{F}\cdot\bar{n}\,dS = \iiint_E \operatorname{div}(F)\,dV$

$= 0$ since $\operatorname{div}(F) = \frac{\partial}{\partial x}(y^3 z) + \frac{\partial}{\partial y}(x^3 z) + \frac{\partial}{\partial z}(1 + e^{x^2+y^2}) = \underline{\underline{0}}$.

Thus, the flux of interest is $\iint_S \bar{F}\cdot\bar{n}\,dS = -\int_{S_1} \bar{F}\cdot\bar{n}\,dS$

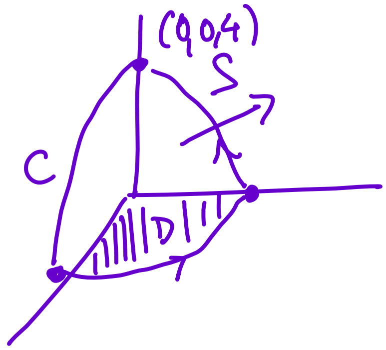
$= -\iint_{S_1} \langle \ldots, \ldots, 1 + e^{x^2+y^2} \rangle \cdot \langle 0, 0, -1 \rangle\,dS = \iint_{\{x^2+y^2 \leq 1\}} 1 + e^{x^2+y^2}\,dA$

$= \int_0^{2\pi}\int_0^1 1 + e^{r^2}\, r\,dr\,d\theta = \boxed{e\pi}$.

in polar

14. (7 points) Let $\mathbf{F}(x,y,z) = \langle yz, -xz, 1\rangle$. Let S be the portion of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant $x \geq 0, y \geq 0, z \geq 0$; let C be the closed curve $C = C_1 + C_2 + C_3$ where the curves C_1, C_2, C_3 are formed by intersecting S with the xy, yz, and xz planes respectively, so that C is the boundary of S. Orient C so that it is traversed counterclockwise when seen from above in the first octant.

Use Stokes' theorem to compute the line integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$ by reducing it to an appropriate surface integral over S.



By Stokes' thm,

$$\oint_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}(\bar{F})\cdot \bar{n}\, dS.$$

We have:

$$\text{Curl}(\bar{F}) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & 1 \end{vmatrix} = \langle x, y, -2z\rangle$$

and $\bar{r}(x,y) = \langle x, y, 4-x^2-y^2\rangle$, $(x,y)\in D = \{x^2+y^2 \leq 4, x \geq 0, y \geq 0\}$

$$\bar{r}_x \times \bar{r}_y = \langle -z_x, -z_y, 1\rangle = \langle 2x, 2y, 1\rangle$$

So $$\iint_S \text{curl}(F)\cdot \bar{n}\, dS = \iint_D \langle x, y, -2(4-x^2-y^2)\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$$

$$= \iint_D 2(x^2+y^2) - 2(4-x^2-y^2)\, dA = 4\int_0^{\pi/2}\int_0^2 (r^2-2)\, r\, dr\, d\theta = 4\cdot\frac{\pi}{2}\int_0^2 r^3 - 2r\, dr$$

$$= 2\pi\left(\frac{2^4}{4} - \frac{2\cdot 4}{2}\right) = \boxed{0}$$

15. (7 points) Let S be the unit sphere centered at the origin, oriented outwards with normal vector $\mathbf{n}$, and let $f(x, y, z) = x + y^2 + z^3$. Calculate

$$\int\int_S D_{\mathbf{n}} f \, dS,$$

where $D_{\mathbf{n}}$ is the directional derivative along $\mathbf{n}$.

We have $D_{\bar{n}} f = \bar{n} \cdot \nabla f$ so the integral is

$$\iint_S \nabla f \cdot \bar{n} \, dS$$, i.e. the flow of ∇f across S.

By the divergence theorem, this integral is equal to:

$$\iiint_E \operatorname{div}(\nabla f) \, dV = \iiint_E \frac{\partial}{\partial x^2} x + \frac{\partial}{\partial y^2} y^2 + \frac{\partial}{\partial z^2} z^3 \, dV$$

Solid sphere

$$= 2\iiint_E dV + 6\iiint_E z \, dV = 2 \operatorname{Vol}(E) + 0$$

center of mass is the origin

$$= \boxed{2 \cdot \frac{4}{3}\pi}$$