# Math 53 Final Exam, Prof. Srivastava
# May 11, 2018, 11:40pm–2:30pm, 155 Dwinelle Hall.

Name: ______________________________

SID: ______________________________

GSI: ______________________________

Name of the student to your left: ______________________________
Name of the student to your right: ______________________________

Instructions: Write all answers clearly in the provided space. This exam includes some space for scratch work at the bottom of pages 2 and 6 which will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit. You are allowed to bring one *single-sided handwritten letter size* cheat sheet. Calculators, phones, textbooks, and your own scratch paper are not allowed. **If you are seen writing after time is up, you will lose 20 points.**

When you are done, hand over your exam to your GSI *unless your GSI is Shiyu Li*, in which case hand it over to me.

UC Berkeley Honor Code: *As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.*

Sign here: ______________________________

| Question: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Points: | 12 | 6 | 6 | 8 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 7 | 7 | 100 |

Do not turn over this page until your instructor tells you to do so.

1. (12 points) Circle always true (**T**) or sometimes false (**F**) for each of the following. There is no need to provide an explanation. Two points each.

   (a) Suppose $f(x, y)$ is differentiable and $f_x = 1$ and $f_y = -2$ at a point. Then there is a direction $u$ such that $D_u f = 0$ at that point. **T** **F**

   (b) If the level curve of a differentiable function $g(x, y) = k$ intersects itself non-tangentially at a point $P$,
   then $P$ must be a critical point of $g$. **T** **F**

   (c) If $a$ and $b$ are vectors in $\mathbb{R}^3$ then $a \times (a \times b)$ is always zero. **T** **F**

   (d) The flux of $F = \langle x, 0, 0 \rangle$ across a sphere of radius 1 at the origin is strictly less than its flux across a sphere of radius 2 at the origin,
   where both are outwardly oriented. **T** **F**

   (e) If $\operatorname{curl}(\nabla f) = \nabla f$ for a function $f$ defined on $\mathbb{R}^3$, then $f$ must be a solution of the PDE
   $$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 + \partial^2 f/\partial z^2 = 0.$$
   **T** **F**

   (f) If $S = \{(x, y, z) : f(x, y, z) = k\}$ is a level surface of a smooth function $f$ with no critical points on $S$, then $S$ must be orientable. **T** **F**

   [Scratch Space Below]

2. Determine whether each of the following statements is true. If so, explain why, and if not, provide a counterexample.

   (a) (3 points) If **F** and **G** are conservative vector fields defined on $\mathbb{R}^3$ then the sum $\mathbf{F} + \mathbf{G}$ is also conservative.

   (b) (3 points) If $\mathbf{F} = \langle P, Q, R \rangle$ and $\mathbf{G} = \langle S, T, U \rangle$ are conservative vector fields defined on $\mathbb{R}^3$, then the vector field

$$\mathbf{H} = \langle PS, QT, RU \rangle$$

   with components equal to their entrywise products, is also conservative.

3. (6 points) A particle moves along the intersection of the surfaces:

$$z = x^2 + \frac{y^2}{4}$$

and

$$x^2 + y^2 = 25.$$

Suppose its position vector at time $t$ is $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ and we know that $x(0) = 3, y(0) = 4$, and $x'(0) = 4$. Calculate $y'(0)$ and $z'(0)$.

4. Suppose $f(x, y) = xy$ and $x = r\cos\theta, y = r\sin\theta$.

   (a) (4 points) Use the chain rule to find the partial derivatives $\partial f/\partial r$ and $\partial f/\partial \theta$.

   (b) (4 points) Use this to approximate the value of $f$ at the point $(r, \theta) = (1.001, -0.01)$.

5. (6 points) Suppose $z = z(x, y)$ is a differentiable function satisfying $e^z = xyz$. Find $\partial z/\partial x$ and $\partial^2 z/\partial x^2$ as functions of $x, y, z$.

[Scratch Space Below]

6. (6 points) Consider the function $f(x, y) = x^3/3 + y^3/3 + 5x - y$. Find and classify the critical points of the function $g(x, y) = |\nabla f(x, y)|^2$.

7. (6 points) Find the extreme values of $f(x,y) = e^{-xy}$ in the region $D = \{(x,y) : x^2 + 4y^2 \leq 1\}$.

8. (6 points) Compute the area of the simply connected "moustache" region enclosed by the parameterized curve

$$\mathbf{r}(t) = \langle 5\cos(t), \sin(t) + \cos(4t) \rangle, \quad t \in [0, 2\pi].$$

9. (6 points) Evaluate the integral

$$\int\int_R e^{x+y} dA$$

where $R$ is given by the inequality $|x| + |y| \leq 1$ by making an appropriate change of variables. (hint: sketch the region first)

10. (6 points) Find the volume of the solid that lies between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 2$.

11. (6 points) Find the work done by the force field $\mathbf{F} = \langle z^2, x^2, y^2 \rangle$ on a particle moving along the line segment from $(1, 0, 0)$ to $(4, 1, 2)$.

12. (6 points) A surface $S$ is parameterized by

$$\mathbf{r}(u,v) = e^{-u^2}\langle 1, \sin(v), \cos(v)\rangle,$$

where

$$0 \leq u \leq \sqrt{\pi}, \quad u^2 \leq v \leq \pi.$$

Find its surface area.

13. (6 points) Find the flux of the vector field

$$\mathbf{F}(x,y,z) = \langle y^3 z, x^3 z, 1 + e^{x^2+y^2} \rangle$$

through the paraboloid part $S$ of the boundary of the solid region

$$z + x^2 + y^2 \leq 1; \quad z \geq 0,$$

where $S$ is oriented upwards.

14. (7 points) Let $\mathbf{F}(x, y, z) = \langle yz, -xz, 1\rangle$. Let $S$ be the portion of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant $x \geq 0, y \geq 0, z \geq 0$; let $C$ be the closed curve $C = C_1 + C_2 + C_3$ where the curves $C_1, C_2, C_3$ are formed by intersecting $S$ with the $xy, yz$, and $xz$ planes respectively, so that $C$ is the boundary of $S$. Orient $C$ so that it is traversed counterclockwise when seen from above in the first octant.

    Use Stokes' theorem to compute the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by reducing it to an appropriate surface integral over $S$.

15. (7 points) Let $S$ be the unit sphere centered at the origin, oriented outwards with normal vector $\mathbf{n}$, and let $f(x, y, z) = x + y^2 + z^3$. Calculate

$$\int\int_S D_{\mathbf{n}} f \, dS,$$

where $D_{\mathbf{n}}$ is the directional derivative along $\mathbf{n}$.