Mathematics H104 A. Ogus
Fall, 2004

Final Solutions

1. (25pts) Define the following terms. Be as precise as you can.

   (a) (3pts) An uncountable set.
   An uncountable set is a set which can not be put into bijection with a finite set or with the set of natural numbers.

   (b) (3pts) A topological space.
   A topological space is a set X together with a set T of subsets of X with the following properties: X and $\emptyset$ belong to T, the union of any family of elements of T belongs to T, and the intersection of any finite number of elements of T belongs to T.

   (c) (3pts) A compact topological space.
   A topological space is compact if every open cover contains a finite subcover.

   (d) (3pts) A connected topological space.
   A topological space is connected if the only subsets of X which are both open and closed are X and $\emptyset$.

   (e) (3pts) A limit point of a subset of a metric space. If E is subset of X and $x \in X$, then x is a limit point of E if every neighborhood of x meets $E \setminus \{x\}$.

   (f) (3pts) A continuous function from X to Y, where X and Y are topological spaces.
   A function $f\colon X \to Y$ is continuous if the inverse image of every open subset of Y is open in X.

   (g) (4pts) Suppose that $f\colon E \to Y$, where $E \subseteq X$ and X and Y are topological spaces, and $x \in X$. What is the definition of

   $$\lim_{e \to x} f(e),$$

   and under what conditions is it unique?
   A point y is a limit if for every neighborhood V of y, there exists a neighborhood U of x such that $U \cap E \setminus \{x\} \subseteq f^{-1}(V)$. Often one requires that x be a limit point of E. It is unique if this is the case and Y is Hausdorff,

   (h) (3pts) An equicontinuous family of real-valued functions on a metric space.
   A family F is equicontinuous if for every $\epsilon > 0$, there exists a δ such that $|f(x) - f(x')| < \epsilon$ whenever $d(x, s') < \delta$ and $f \in F$.

2. (20pts) Give an example of the following, or explain, using results discussed in class or the book, why no such example exists. If you give an example, you need not prove that it works.

(a) (5pts) A continuos real-valued function on a bounded subset of the reals which is unbounded.
The function $1/x$ on $(0, 1)$.

(b) (5pts) A uniformly continuous function on a bounded subset of the reals which is unbounded.
This can't exist. If f is uniformly continuous there exists a $\delta > 0$ such that $|f(x) - f(x')| < 1$ whenever $|x - x'| < \delta$. Then if $|x - x'| < k\delta$, $|f(x) - f(x')| < k$. Since the domain of f is bounded, it follows that there exists k such that this is true for any pair of points, and it follows that f is bounded.

(c) (5pts) A sequence of continuous real-valued functions on $[0, 1]$ which has no pointwise convergent subsequence.
The sequence $\sin(nx)$ has this property. So does the sequence $f_n = n$.

(d) A uniformly bounded sequence of real-valued continuous functions on $\mathbf{Q} \cap [0, 1]$ which has no pointwise convergent subsequence.
This doesn't exist, by a rather tricky diagonalization argument.

3. (10pts) Prove that the closed interval $[0, 1]$ is compact.
Let $\mathcal{U}$ be an open cover of $[0, 1]$. Let S be the set of all $x \in [0, 1]$ such that $[0, x]$ can be covered by a finite subset of $\mathcal{U}$. Clearly $0 \in S$, and S is bounded, hence has a supremum, s. Evidently $s \leq 1$. We claim that $s \in S$. If $s = 0$ this is clear. If $s > 0$, there is a $U \in \mathcal{U}$ such that $s \in U$, and there is an $s' \in [0, s)$ such that $[s', s] \subseteq U$. Since $s' < s$, there is an $s'' > s'$ with $s'' \in S$, and hence a finite subset $\mathcal{U}'$ of $\mathcal{U}$ which covers $[0, s'']$. Then $\mathcal{U}'$ together with U covers $[0, s]$, and $s \in S$. Furthermore, if $s < 1$, we see that U also contains an interval of the form $[s, t]$ with $t \in (s, 1]$, so that we also have a finite cover of $[0, t]$. Hence $s = 1$, and we are done.

4. (10pts) Let A be a nonempty closed subset of a metric space X. For $x \in X$, let $d_A(x) := \inf\{d(x, a) : a \in A\}$.

(a) Explain why the above definition makes sense.
This is because the set in question is bounded below and nonempty. Hence it has an infimum.

(b) Prove that d_A is continuous, and that $x \in A$ iff $d_A(x) = 0$.
Suppose that x and y are points of X. Then for any $a \in A$, $d_A(x) \leq d(x, a) \leq d(x, y) + d(y, a)$. Thus $d_A(x) - d(x, y)$ is a lower bound for $\{d(y, a) : a \in A)\}$ and hence $d_A(y) \geq d_A(x) - d(x, y)$. In other words, $d_A(x) - d_A(y) \leq d(x, y)$. By symmetry, the same is true with x and y interchanged, and we conclude that $|d_A(x) - d_A(y)| \leq d(x, y)$. This shows that d_A is uniformly continuous. Finally, if $x \in A$, $d_A(x) \leq d(x, x) = 0$, and if $x \notin A$, since A is closed there is an $\epsilon > 0$ such that $B_\epsilon(x) \cap A = \emptyset$. Hence $d(x, a) \geq \epsilon$ for all a and $d_A(x) \geq \epsilon$.

(c) Prove that if A and B are disjoint closed subsets of X, then there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Hint: consider $\{x : d_A(x) < d_B(x)\}$.

It follows from the above that this set U is open and contains A, since if $a \in A$, $d_A(x) = 0$ and $d_B(x) > 0$. On the other hand, if $V := \{x : d_B(x) < d_A(x)\}$, V is open and contains B. Clearly the two sets are disjoint.

5. (10pts) Prove that if $0 < r < 1$, nr^n approaches 0 as n approaches infinity. You get more credit for an elementary proof. You may find it easier to look at the reciprocal.
It suffices to prove that if $r > 1$, r^n/n approaches infinity as n approaches infinity. Write $r = 1 + x$, where $x > 0$. Then

$$r^n/n \geq n^{-1}\left(1 + nx + \frac{n(n-1)}{2}x^2\right) \geq (n-1)x^2/2,$$

which clearly goes to infinity.

6. (10pts) Let $f\colon \mathbf{R} \to \mathbf{R}$ be a differentiable function. Suppose that $f'(x) \neq 1$ for all $x \in \mathbf{R}$. Prove that there is at most one number a such that $f(a) = a$.
If a and b are fixed points of f and $a < b$, then by the MVT there exists an $x \in (a, b)$ such that $f'(x)(b - a) = f(b) - f(a) = b - a$. But then $f'(x) = 1$, a contradiction.

7. (10pts) Suppose $f(0) = f'(0) = f''(0) = 0$. Prove that if $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x)| \leq \epsilon |x|^2$ whenever $|x| < \delta$.
First observe that since $f'(0) = 0$,

$$0 = f''(0) = \lim_{x \to 0} \frac{f'(x)}{x}.$$

Suppose $\epsilon > 0$ is given. Then there exists a $\delta > 0$ such that $|f'(x)| \leq \epsilon |x|$ whenever $|x < \delta$. Furthermore, since $f(x) = 0$, for any such x, the MVT says that there is some $x' \in (0, x)$ such that $f'(x')x = f(x)$. Hence $|f(x) \leq \epsilon |x'||x| \leq \epsilon |x|^2$.

8. (15pts) Consider the series $\sum z^n/n$, where z is a complex number.

   (a) Prove that the series converges pointwise on $[-1, 1)$. The first statement follows from Abel's theorem. For each m, if $z \neq 1$ and $z \in [-1, 1)$,

   $$|A_m| := \left|\sum_1^m z^n\right| = \left|\frac{z}{1-z} - \frac{z^{m+1}}{1-z}\right| \leq \frac{2}{|1-z|}.$$

   Thus the sequence of partial sums is bounded and hence Abel's theorem applies.

   (b) Prove that the convegence is uniform on $[-1, 1-\epsilon)$ for every $\epsilon > 0$. For the uniformity we have to recall the proof. If $\epsilon > 0$, then there exists a number $M > 0$ such that for all $z \in [-1, \epsilon]$, $|A_m| \leq M$. Now if $q > p > 1$,

   $$\left|\sum_p^q \frac{z^n}{n}\right| = \left|\sum_p^q \frac{A_n - A_{n-1}}{n}\right| = \left|\sum_p^q \frac{A_n}{n} - \sum_{p-1}^{q-1} \frac{A_n}{n+1}\right|$$

$$= \left| \frac{A_q}{q} - \frac{A_{p-1}}{p} + \sum_p^{q-1} A_n \left(\frac{1}{n} - \frac{1}{n+1} \right) \right|$$
$$\leq \left| \frac{A_q}{q} \right| + \left| \frac{A_{p-1}}{p} \right| + \sum_p^{q-1} |A_n| \left| \frac{1}{n} - \frac{1}{n+1} \right|$$
$$\leq \frac{M}{q} + \frac{M}{p} + \sum_p^{q-1} M \left(\frac{1}{n} - \frac{1}{n+1} \right) \leq \frac{2M}{p}.$$

Thus the sequence is uniformly Cauchy and hence uniformly convergent.

Alternatively, we can use a simpler argument, based on alternating series. First observe that if $r \in (0,1)$, the series converges absolutely and uniformly in any interval of the form $[-r, r]$, by the ratio test, for example. So it suffices to prove uniform convergence in the interval $[-1, 0]$. Then the usual alternating series trick shows that if $z \in [-1, 0]$,

$$\sum_N^M \frac{z^n}{n} < \left| \frac{z^{N+1}}{N+1} \right| < \frac{1}{N+1}.$$

This shows that the series is uniformly Caucy, hence uniformly convergent.

(c) What can you deduce about the function $f(z) = \sum z^n/n$ for $z \in [-1, 1)$? Can you find and prove a simple formula for it, valid at all these points? In particular, what is its value at -1?

It follows from the uniform convegence that the limit function $f(z)$ is continuous on $[-1, 1)$. Now we know that $1 + z + z^2 - \cdots$, converges to $(1-z)^{-1}$, and the convergence is uniform on compact subsets of $(-1, 1)$. Hence we can integrate term by term, and deduce that $z - z^2/2 + \cdots$ converges to $-\log(1-z)$ on the open interval $(-1, 1)$. Hence $f(z) = -\log z$ if $|z| < 1$. But both functions are also continuous at -1, hence they agree there also. Thus when $z = -1$, the series converges to $-\log 2$.

9. (10pts) Prove that if $f\colon X \to Y$ is a continuous bijective map of compact Hausdorff spaces, then the inverse mapping $Y \to X$ is also continuous. Show that this need not be the case if X is not compact.

Let $g\colon Y \to X$ be the inverse mapping, and let Z be a closed subset of X. Then $g^{-1}(Z) = f(Z)$. But Z is compact, hence $f(Z)$ is compact, hence closed. Thus g^{-1} takes closed sets to closed sets, so g is continuous.

On the other hand, let $X := [0, 2\pi)$, let $Y := \{z \in \mathbf{C} : |z| = 1\}$, and let $f\colon X \to Y$ the be map sending θ to $\cos(\theta) + i \sin(\theta)$. Then f is continuous and bijective. However its inverse g is not continuous, since the inverse image of every neighborhood of $1 \in Y$ contains points near 0 and near 2π.

10. (15pts) Let $C(T)$ denote the set of continuous functions $\mathbf{R} \to \mathbf{C}$ which are periodic with period 2π. Let $\mathcal{A} \subseteq C(T)$ denote the set of finite linear combinations of functions of the form e^{inx} with $n \in \mathbf{Z}$. If f and g belong to $C(T)$, let $(f \star g)(x) := \int_{-\pi}^{\pi} f(x-t)g(t)dt$.

(a) Show that if $g \in \mathcal{A}$, then $f \star g \in \mathcal{A}$.
Let $s = x - t$, and observe that

$$(f \star g)(x) = \int_{-\pi}^{\pi} f(x-t)g(t)dt = -\int_{-\pi+x}^{x-\pi} f(s)g(x-s)ds = \int_{\pi-x}^{\pi+x} f(s)g(x-s)ds$$

Since f and g are periodic with period 2π,

$$\int_{\pi-x}^{\pi+x} f(s)g(x-s)ds = \int_{\pi}^{\pi} f(s)g(x-s)ds.$$

Now if $g_n(t) := e^{int}$, $g(x-s) = g(x)g(-s)$, and hence

$$(f \star g_n)(x) = \int_{-\pi}^{\pi} f(s)g(x)g(-s)ds = g_n(x)(f \star g)(0).$$

This is an element of $\mathcal{A}$. Hence the same is true if g is a linear combination of the g_n's.

(b) Let

$$c_n := \int_{-\pi}^{\pi} \left(\frac{1+\cos t}{2}\right)^n dt = 2\int_0^{\pi} \left(\frac{1+\cos t}{2}\right)^n dt$$

Show that

$$c_n \geq 2\int_0^{\pi} \left(\frac{1+\cos t}{2}\right)^n \sin t dt.$$

Use this last inequality to show that $c_n \geq c/(n+1)$ for some c .
Let

$$Q_n(t) := c_n^{-1} \left(\frac{1+\cos t}{2}\right)^n$$

and prove if $\delta > 0$, Q_n converges uniformly to 0 on $[\delta, \pi]$.
Since $\sin(t) \leq 1$,

$$c_n \geq 2\int_0^{\pi} \left(\frac{1+\cos t}{2}\right)^n \sin t dt = 2\frac{(1+\cos t)^{n+1}}{2^n(n+1)}|_0^{\pi} = \frac{4}{n+1}.$$

Now if $\delta > 0$ and $t \in [\delta, \pi]$, $\left(\frac{1+\cos t}{2}\right) \leq r$, where $r \in (0,1)$. Hence $Q_n(t) \leq M(n+1)r^n$ for some M, which approaches zero.

(c) Deduce that $\mathcal{A}$ is uniformly dense in $C(T)$. You may use the theorem on Dirac families.
The above arguments shows that the set of Q_n form a Dirac family. Hence for any continuous f, $f \star Q_n$ converges uniformly to f.