Mathematics H104 A. Ogus

Fall, 2004

Midterm II Solutions

1. Suppose that f is a differentiable function on $[0,1]$ such that f' never vanishes and $f(0) = 0$ and $f(1) = 1$. Prove that f is strictly increasing.

   By the intermediate value theorem for derivatives, f' is always positive or always negative. If were always negative, f would be strictly decreasing, by the mean value theorem, contradciting the hypothesis. Hence f' is postiive and f is increasing.

2. Let $f\colon[-1,1] \to \mathbf{R}$ be a bounded function which is integrable with respect to the function α, where $\alpha(x) = 0$ if $x \leq 0$ and $\alpha(x) = 1$ if $x > 0$. Is f necessarily continuous at 0? Proof or counterexample (with a proof that your example works.)

   No, f need not be continuous. For example, consider the function f such that $f(x) = 0$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. Let P be any partion of $[-1,1]$ containing 0, say $x_{i-1} = 0$. Then $\Delta\alpha_j = 0$ if $i \neq j$, but $m_i = M_i$ since f is constant on I_i. Hence $L(f,P) = U(f,P) = 1$ and f is integrable.

3. Let X and Y be metric spaces and let $f\colon X \to Y$ be a continuous function.

   (a) Prove that f is uniformly continuous if X is compact, directly from the definition of compactness.
   If $\epsilon > 0$, then for each $x \in X$ there is a δ_x such that $d(f(x), f(x')) < \epsilon$ if $d(x', x) \leq \delta_x$. Since X is compact, there is a finite set of x_i such that the set of balls of the form $B_{\delta_{x_i}/2}$ covers X. Let δ be the minimum of these radii and let x and x' be two points of X with $d(x, x') < \delta$. Then there exists some i such that $x \in B_{\delta_{x_i}/2}$ Since $d(x, x') < \delta_{x_i}/2$, $x' \in B_{\delta_{x_i}}$. Hence $d(f(x,)f(x_i)) < \epsilon$ and $d(f(x'), f(x_i)) < \epsilon$, so $d(f(x), f(x') < 2\epsilon$.

   (b) Show by example that f need not be uniformly continuous if X is not compact, even if Y is bounded. You need not prove your example works.
   The function $sin(1/x)$ for $x \in [1, \infty)$ is an example.

4. For which values of $x \in \mathbf{C}$ and $s \in \mathbf{R}$ does the series $\sum x^n n^{-s}$ converge absolutely, converge conditionally, or diverge? Explain and justify each case.

If $|x| < 1$, then for any s, the series converges absolutely. Indeed, if $x = 0$ this is trivial, and if $x > 0$ the ratio of two successive terms is $|x|(1 + 1/n)^s$, which approaches $|x|$. Thus the ratio test applies.

If $|x| > 1$, the series diverges, as the ratio test above shows.

If $|x| = 1$ and $s \leq 0$, the nth term doesn't approach zero, so the series diverges.

If $x = 1$ and $x \in (0, 1)$, Cauchy's test applies. Namely, $\sum_k 2^k (2^{-ks})$ becomes a geometric series with ratio 2^{1-s}, which diverges if $s \leq 1$ and converges if $s > 1$, so the same is true of the original series.

Suppose $|x| = 1$ but $x \neq 1$. Then the series converges absolutely if $s > 1$ by the previous part. It remains only to prove that it converges (hence conditionally) if $s \in (0, 1]$. This follows from Abel's theorem. Namely, the sequence of partial sums of the series $\sum x^n$ is bounded, and the sequence n^{-s} goes to zero.