Name:____

Student ID Number:________________________________

Instructions:

1. You will have 80 minutes to complete the exam.
2. The exam is a total of 5 questions and each question is worth 10 points.
3. Unless stated otherwise, you may use results we proved in class and on the homework.
4. No books, notes, calculators, or electronic devices are permitted.
5. If you require additional space, please use the reverse side of the pages.
6. The exam has a total of 7 pages with the last page left blank for scratch work. Please verify that your copy has all 7 pages.

Question	**Score**	**Points**
1.		10
2.		10
3.		10
4.		10
5.		10
Total		50

1. (a) For (E, d) a metric space, $(x_n)_{n\in\mathbb{N}} \subset E$ a sequence, and $x \in E$ a point, state what it means for the sequence $(x_n)_{n\in\mathbb{N}}$ to **converge** to x.

   (b) In the metric space $(\mathbb{R}, d)$, where $d(x, y) = |x - y|$, find the limit of the following sequence and prove it converges: $x_n = \dfrac{2n^3 + 6}{n^3 + 2n + 1}$, $n \in \mathbb{N}$.

2. (a) Prove $\inf \left\{ \frac{1}{10^n} : n \in \mathbb{N} \right\} = 0$.

   (b) Consider the following infinite decimal expansion:

$$0.999\cdots = \sup\{0.\underbrace{9\cdots9}_{n \text{ digits}} : n \in \mathbb{N}\}.$$

   Prove $0.999\cdots = 1$.

3. (a) For a set E and a map $d\colon E \times E \to \mathbb{R}$, state the **four** conditions required for a (E, d) to be a metric space.

   (b) Consider the set of sequences of real numbers between 0 and 1:

$$E = \left\{(a_n)_{n\in\mathbb{N}} \colon \forall n \in \mathbb{N},\ 0 \leq a_n \leq 1\right\}.$$

   Let $d\colon E \times E \to \mathbb{R}$ be the function defined by

$$d\left((a_n)_{n\in\mathbb{N}},\, (b_n)_{n\in\mathbb{N}} \right) := \sup_{n\in\mathbb{N}} |a_n - b_n|.$$

   Prove that (E, d) is a metric space.

4. (a) In a metric space (E, d), state what it means for a subset $S \subset E$ to be **open**.
   (b) In a metric space (E, d), state what it means for a subset $S \subset E$ to be **closed**.
   (c) With (E, d) as in Question 3.(b), show that the following subset of E is closed:

$$S := \left\{ (a_n)_{n \in \mathbb{N}} \in E \colon \lim_{n \to \infty} a_n = 0 \right\}.$$

5. Let $\mathbb{Q} = \{q_n\}_{n\in\mathbb{N}}$ be any enumeration of the rational numbers.

   (a) For arbitrary $x \in \mathbb{R}$, prove that there exists a subsequence $(q_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} q_{n_k} = x$.
   [**Warning:** remember that a subsequence must have increasing indices $n_1 < n_2 < n_3 < \cdots$.]

   (b) Prove whether or not the sequence $(q_n)_{n\in\mathbb{N}}$ converges.

Scratch Work.