1. (a) **(3 pts)** The function f is a C^r-diffeomorphism if it is a bijection, of class C^r, and its inverse is of class C^r.

   (b) **(7 pts)** We first note that f is indeed a bijection since its inverse is given by

$$g(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

The the coordinate functions of f and g are polynomials (degree one polynomials, but polynomials nonetheless), which are smooth. By a theorem from class, this implies f and g are smooth functions. In particular, they are of class C^r for every $r \in \mathbb{N}$. Hence f is a C^r-diffeomorphism. □

2. (a) **(3 pts)** S is Riemann measurable if its characteristic function χ_S is Riemann integrable. Equivalently, if ∂S is a zero set.

   (b) **(7 pts)** Let $T_1, T_2 \subset R$ be the triangles

$$T_1 := \{(x, y) \in R \colon x \leq y\}$$
$$T_2 := \{(x, y) \in R \colon x > y\}.$$

Since their boundaries are composed of line segments, which are zero sets, these sets are Riemann measurable. Consequently χ_{T_1} and χ_{T_2} are Riemann integrable. Observe that $f = a\chi_{T_1} + b\chi_{T_2}$, thus f is Riemann integrable and

$$\int_R f = a \int_R \chi_{T_1} + b \int_R \chi_{T_2}.$$

Now, to compute these integrals we invoke Fubini's theorem:

$$\int_R \chi_{T_1} = \int_0^1 \int_0^1 \chi_{T_1}(x, y)\ dxdy = \int_0^1 \int_0^y 1\ dxdy = \int_0^1 y\ dy = \left.\frac{y^2}{2}\right|_0^1 = \frac{1}{2}.$$

For T_2 we note $\chi_{T_2} = \chi_R - \chi_{T_1}$ and thus

$$\int_R \chi_{T_2} = \int_R \chi_R - \int_R \chi_{T_1} = |R| - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Thus $\int_R = \frac{a}{2} + \frac{b}{2}$. □

3. (a) **(3 pts)** A smooth function $\varphi \colon [0,1]^k \to \mathbb{R}^n$ is a k-cell in $\mathbb{R}^n$.

   (b) **(7 pts)** We first note

$$\frac{\partial\varphi_{(1,2)}(u_1, u_2)}{\partial u} = \det \begin{bmatrix} \cos(2\pi u_2) & -2\pi u_1 \sin(2\pi u_2) \\ \sin(2\pi u_2) & 2\pi u_1 \cos(2\pi u_2) \end{bmatrix} = 2\pi u_1 \cos^2(2\pi u_2) + 2\pi u_1 \sin^2(2\pi u_2) = 2\pi u_1.$$

Thus by Fubini's theorem

$$\begin{aligned} \int_\varphi \omega &= \int_{[0,1]^2} f \circ \varphi(u) \frac{\partial \varphi_{(1,2)}(u)}{\partial u}\ du \\ &= \int_0^1 \int_0^1 u_1 \sin(2\pi u_2) 2\pi u_1\ du_1 du_2 \\ &= 2\pi \int_0^1 u_1^2\ du_1 \int_0^1 \sin(2\pi u_2)\ du_2 \\ &= 2\pi \left[\frac{u_1^3}{3}\right]_0^1 \left[\frac{-\cos(2\pi u_2)}{2\pi}\right]_0^1 = 0. \end{aligned}$$

□

[Alternative Proof:] Using the change of variables $r = u_1$ and $\theta = 2\pi u_2$ we have

$$\int_\varphi \omega = \int_0^{2\pi} \int_0^1 f(r\cos(\theta), r\sin(\theta)) r\ drd\theta.$$

By a homework exercise, this equals $\int_S f$ where

$$S = \{(x,y) \in \mathbb{R}^n \colon x^2 + y^2 \leq 1\}.$$

However, since $f(x,-y) = -y = -f(x,y)$, the symmetry of S across the x-axis implies $\int_S f = 0$. □

4. (a) **(3 pts)**

$$\begin{aligned} \text{wedge product} \quad & \wedge \colon \Omega^k(\mathbb{R}^n) \times \Omega^\ell(\mathbb{R}^n) \to \Omega^{k+\ell}(\mathbb{R}^n) \\ \text{exterior derivative} \quad & d \colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n) \\ \text{pullback} \quad & T^*\Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n). \end{aligned}$$

(b) **(7 pts)** Using the definition of the exterior derivative:

$$d\omega = d(f_1) \wedge dy_{(1,2)} + d(f_2) \wedge dy_{(1,3)} + d(f_3) \wedge dy_{(2,3)}.$$

Now, by a theorem from lecture we know that for $f \colon \mathbb{R}^3 \to \mathbb{R}$ smooth we have

$$d(f) = \frac{\partial f}{\partial y_1} dy_1 + \frac{\partial f}{\partial y_2} dy_2 + \frac{\partial f}{\partial y_3} dy_3.$$

So continuing our previous computation and using signed commutativity we have

$$\begin{aligned} d\omega &= (2y_1 dy_1 + 0dy_2 + y_1^2 dy_3) \wedge dy_{(1,2)} + (0dy_1 + y_3 dy_2 + y_2 dy_3) \wedge dy_{(1,3)} + (y_3 dy_1 + 0dy_2 + y_1 dy_3) \wedge dy_{(2,3)} \\ &= 2y_1 dy_{(1,1,2)} + y_1^2 dy_{(3,1,2)} + y_3 dy_{(2,1,3)} + y_2 dy_{(3,1,3)} + y_3 dy_{(1,2,3)} + y_1 dy_{(3,2,3)} \\ &= 0 + (-1)^2 y_1^2 dy_{(1,2,3)} + (-1)^1 y_3 dy_{(1,2,3)} + 0 + y_3 dy_{(1,2,3)} + 0 \\ &= y_1^2 dy_{(1,2,3)}. \end{aligned}$$

□