**Name:**________________________________________

**Student ID Number:**______________________________

**Instructions:**

1. You will have 50 minutes to complete the exam.
2. The exam is a total of 4 questions and each question is worth 10 points.
3. Unless stated otherwise, you may use results we proved in class and on the homework.
4. No books, notes, calculators, or electronic devices are permitted.
5. If you require additional space, please use the reverse side of the pages.
6. The exam has a total of 6 pages with the last page reserved for scratch work. Please verify that your copy has all 6 pages.

| **Question** | **Score** | **Points** |
|---|---|---|
| 1. | | 10 |
| 2. | | 10 |
| 3. | | 10 |
| 4. | | 10 |
| **Total** | | 40 |

1. (a) Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be a function. For $r \in \mathbb{N}$, state what it means for $f$ to be a $C^r$-**diffeomorphism**.

   (b) Fix $n \in \mathbb{N}$ and let $\sigma \in S_n$ be a permutation on $\{1, 2, \ldots, n\}$. Show that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

   is a $C^r$-diffeomorphism for every $r \in \mathbb{N}$.

2. (a) For $S \subset \mathbb{R}^2$, say what it means for $S$ to be **Riemann measurable**.

   (b) Let $R = [0,1] \times [0,1] \subset \mathbb{R}^2$ and fix $a, b \in \mathbb{R}$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) := \begin{cases} a & \text{if } x \leq y \\ b & \text{otherwise} \end{cases}.$$

   Show that $f$ is Riemann integrable on $R$ and compute $\int_R f$.

3. (a) For $k, n \in \mathbb{N}$, give the definition of a $k$-**cell in** $\mathbb{R}^n$.

   (b) Let $\varphi \in C_2(\mathbb{R}^2)$ be defined by

   $$\varphi(u_1, u_2) = (u_1 \cos(2\pi u_2), u_1 \sin(2\pi u_2)).$$

   Let $\omega = f dy_{(1,2)} \in \Omega^2(\mathbb{R}^2)$ with $f(y_1, y_2) = y_2$. Compute $\int_\varphi \omega$.

4. (a) Let $k, \ell, n, m \in \mathbb{N}$ and let $T \colon \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map. Give the name of each map and fill in the blanks:

| Name of map: | Fill in the blanks: |
|---|---|
| ________________ | $\wedge \colon \Omega^k(\mathbb{R}^n) \times \Omega^\ell(\mathbb{R}^n) \to \Omega^{\square}(\mathbb{R}^n)$ |
| ________________ | $d \colon \Omega^k(\mathbb{R}^n) \to \Omega^{\square}(\mathbb{R}^n)$ |
| ________________ | $T^* \colon \Omega^k(\mathbb{R}^{\square}) \to (\mathbb{R}^{\square})$ |

(b) For

$$\omega = f_1 dy_{(1,2)} + f_2 dy_{(1,3)} + f_3 dy_{(2,3)} \in \Omega^2(\mathbb{R}^3),$$

with

$$f_1(y_1, y_2, y_3) = y_1^2 y_3$$
$$f_2(y_1, y_2, y_3) = y_2 y_3$$
$$f_3(y_1, y_2, y_3) = y_1 y_3$$

compute the **ascending presentation** of $d\omega$.

**Scratch Work:**