1. (a) **(3 pts)** The operator norm is the quantity

$$\|T\| := \left\{ \frac{|Tv|}{|v|} : v \in V \setminus \{0\} \right\}.$$

(b) **(7 pts)** For $f \in C([0,1])$ we have

$$|\delta_{(x,y)}(f)| = |(f(x), f(y))| \leq \sqrt{f(x)^2 + f(y)^2} \leq \sqrt{\|f\|_\infty^2 + \|f\|_\infty^2} = \sqrt{2}\|f\|_\infty.$$

Hence $\|\delta_{(x,y)}\| \leq \sqrt{2}$. On the other hand, if $f(x) = 1$ is a constant function. Then $\|f\|_\infty = 1$ and hence

$$\|\delta_{(x,y)}\| \geq |\delta_{(x,y)}(f)| = |(1,1)| = \sqrt{2}.$$

Thus $\|\delta_{(x,y)}\| = \sqrt{2}$. □

2. (a) **(3 pts)** R is sublinear if

$$\lim_{v \to 0} \frac{R(v)}{|v|} = 0.$$

(b) **(7 pts)** We claim

$$(Df)_p = \begin{bmatrix} p_2 & p_1 \\ 1 & 2p_2 \\ 1 & 1 \end{bmatrix}.$$

The corresponding Taylor remainder for $v = (v_1, v_2)$ is then

$$\begin{aligned} R(v) &= f(p+v) - f(p) - (Df)_p(v) \\ &= \begin{pmatrix} (p_1+v_1)(p_2+v_2) \\ p_1 + v_1 + (p_2+v_2)^2 \\ p_1 + v_1 + p_2 + v_2 \end{pmatrix} - \begin{pmatrix} p_1 p_2 \\ p_1 + p_2^2 \\ p_1 + p_2 \end{pmatrix} - \begin{pmatrix} p_2 v_1 + p_1 v_2 \\ v_1 + 2p_2 v_2 \\ v_1 + v_2 \end{pmatrix} \\ &= \begin{pmatrix} v_1 v_2 \\ v_2^2 \\ 0 \end{pmatrix}. \end{aligned}$$

Now, if $v_2 = 0$, we have $R(v)/|v| = 0$. Otherwise, using $|v| \geq |v_2|$ we have

$$\frac{|R(v)|}{|v|} \leq \frac{\sqrt{(v_1 v_2)^2 + v_2^4}}{|v_2|} = \sqrt{v_1^2 + v_2^2} = |v|,$$

which clearly tends to zero as $|v|$ does. Hence R is sublinear and our claim holds. □

3. (a) **(3 pts)** T is k-linear if it is linear in each coordinate. That is, for $1 \leq j \leq k$, $v_1, \ldots, v_k \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$ we have

$$T(v_1, \ldots, v_{j-1}, v_j + \alpha w, v_{j+1}, \ldots, v_k) = T(v_1, \ldots, v_k) + \alpha T(v_1, \ldots, v_{j-1}, w, v_{j+1}, \ldots, v_k).$$

(b) **(7 pts)** Using basic differentiation rules, we can compute the second partial derivatives of the component functions of f:

$$\frac{\partial^2 f_1(p)}{\partial x_1^2} = 0 \quad \frac{\partial^2 f_1(p)}{\partial x_1 \partial x_2} = 1 \quad \frac{\partial^2 f_1(p)}{\partial x_2 \partial x_1} = 1 \quad \frac{\partial^2 f_1(p)}{\partial x_2^2} = 0$$

$$\frac{\partial^2 f_2(p)}{\partial x_1^2} = 0 \quad \frac{\partial^2 f_2(p)}{\partial x_1 \partial x_2} = 0 \quad \frac{\partial^2 f_2(p)}{\partial x_2 \partial x_1} = 0 \quad \frac{\partial^2 f_2(p)}{\partial x_2^2} = 2$$

$$\frac{\partial^2 f_3(p)}{\partial x_1^2} = 0 \quad \frac{\partial^2 f_3(p)}{\partial x_1 \partial x_2} = 0 \quad \frac{\partial^2 f_3(p)}{\partial x_2 \partial x_1} = 0 \quad \frac{\partial^2 f_3(p)}{\partial x_2^2} = 0$$

Since these are all clearly continuous, we have that $(D^2 f)_p$ exists. □

[**Alternate Proof:**] We order pairs of basis vectors for $\mathbb{R}^2$ as follows:

$$\{(e_1, e_1), (e_1, e_2), (e_2, e_1), (e_2, e_2)\}.$$

We then claim that $(D^2 f)_p$ has the following matrix representation (the entries coming from the computations in the previous proof):

$$(D^2 f)_p = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding Taylor remainder for $(Df)_p$ at $(v, w) \in \mathbb{R}^2 \times \mathbb{R}^2$ is:

$$\begin{aligned} R(v, w) &= (Df)_{p+v}(w) - (Df)_p(w) - (D^2 f)_p(v, w) \\ &= \begin{pmatrix} (p_2 + v_2)w_1 + (p_1 + v_1)w_2 \\ w_1 + 2(p_2 + v_2)w_2 \\ w_1 + w_2 \end{pmatrix} - \begin{pmatrix} p_2 w_1 + p_1 w_2 \\ w_1 + 2p_2 w_2 \\ w_1 + w_2 \end{pmatrix} - \begin{pmatrix} v_1 w_2 + v_2 w_1 \\ 2v_2 w_2 \\ 0 \end{pmatrix} = 0. \end{aligned}$$

Since this holds for all $w \in \mathbb{R}^2$, we have $\|R(v, \cdot)\| = 0$. Consequently $\lim_{v \to 0} \frac{R(v, \cdot)}{|v|} = 0$. □

4. (a) **(3 pts)** The sequence is uniformly C^r convergent if there exists $f : U \to \mathbb{R}^m$ of class C^r such that $(D^i f_k)_{k \in \mathbb{N}}$ converges uniformly to $D^i f$ on U for each $i = 0, 1, \ldots, r$.

   (b) **(7 pts)** We claim the sequence is uniformly C^1 convergent to $f(x, y) = (x^2, y)$. Indeed, we first note that for $(x, y) \in B$

$$|f_k(x, y) - f(x, y)| = \left| \left(\frac{x}{k}, \frac{1}{k} \right) \right| = \frac{1}{k}\sqrt{x^2 + 1} \leq \frac{1}{k}\sqrt{2}$$

Since this upper bound holds for all $(x, y) \in B$ and tends to zero, we see that $(f_k)_{k \in \mathbb{N}}$ converges uniformly to f on B. Now, from basic differentiation rules we know the partial derivatives of the component functions of f_k and f, and since they are continuous it follows from a theorem proved in lecture these partial derivatives are the entires of the total derivatives. Namely,

$$(Df_k)_{(x,y)} = \begin{bmatrix} 2x + \frac{1}{k} & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad (Df)_{(x,y)} = \begin{bmatrix} 2x & 0 \\ 0 & 1 \end{bmatrix}.$$

Consequently,

$$(Df_k)_{(x,y)} - (Df)_{(x,y)} = \begin{bmatrix} \frac{1}{k} & 0 \\ 0 & 0 \end{bmatrix}.$$

From a homework exercise, we know that the operator norm of this diagonal matrix is $\frac{1}{k}$. Thus $\|(Df)_{(x,y)} - (Df)_{(x,y)}\| = \frac{1}{k}$ for all $(x, y) \in B$. Since this tends to zero, we see that $(Df_k)_{k \in \mathbb{N}}$ converges uniformly to Df. Thus $(f_k)_{k \in \mathbb{N}}$ is uniformly C^1 convergent. □