1. (a) **(4 pts)** The sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in (E,d) if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ so that $\forall n \geq N$, $d(x_n, x) < \epsilon$.

   (b) **(6 pts)** We claim the limit is 3. Let $\epsilon > 0$. Choose $N > \sqrt{3/\epsilon}$. Then for any $n \geq N$ we have $\frac{3}{n^2} \leq \frac{3}{N^2} < \epsilon$ and thus
$$d(x_n, 3) = \left|\frac{3n^3}{n^3+n} - 3\right| = \left|\frac{3n^3 - 3n^3 - 3n}{n^3+n}\right| = \left|\frac{-3n}{n^3+n}\right| = \frac{3n}{n^3+n} \leq \frac{3n}{n^3} = \frac{3}{n^2} < \epsilon.$$
□

2. (a) **(4 pts)** (E,d) is a metric space if for all $x, y, z \in E$
   (i) $d(x,y) \in [0, +\infty)$;
   (ii) $d(x,y) = 0$ iff $x = y$;
   (iii) $d(x,y) = d(y,x)$;
   (iv) $d(x,y) \leq d(x,z) + d(z,y)$.

   (b) **(6 pts)** We check the four parts of the definition from part (a):
   (i) Since $|x-y| \geq 0$ for all $x, y \in \mathbb{R}$ and $1 \geq 0$, we obtain $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$.
   (ii) If $d(x,y) = 0$, then necessarily we have $d(x,y) = |x-y|$. Hence $|x-y| = 0$ which implies $x = y$. Conversely, if $x = y$ then
$$d(x,y) = \min\{|x-y|, 1\} = \min\{0, 1\} = 0.$$
   (iii) Since $|x-y| = |-(x-y)| = |-x+y| = |y-x$, we have
$$d(x,y) = \min\{|x-y|, 1\} = \min\{|y-x|, 1\} = d(y,x).$$
   (iv) We first note that if $d(x,z) + d(z,y) \geq 1$ then we immediately obtain
$$d(x,y) \leq 1 \leq d(x,z) + d(z,y).$$
   Otherwise, $d(x,z) + d(z,y) < 1$ and consequently $d(x,z) < 1$ and $d(z,y) < 1$. This means $d(x,z) = |x-z|$ and $d(z,y) = |z-y|$. But then the triangle inequality for the absolute value implies
$$|x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y) < 1.$$
   Consequently,
$$d(x,y) = \min\{|x-y|, 1\} = |x-y|,$$
   and so the previous inequality implies $d(x,y) \leq d(x,z) + d(z,y)$. □

3. (a) **(3 pts)** A subset S is open if for every $x \in S$ there exists $r > 0$ such that $B(x,r) \subset S$.

   (b) **(7 pts)** Let $(x_0, y_0) \in S$. Then $-5 < x_0 < 5$, and so if we set
$$r := \min\{5 - x_0, x_0 + 5\}$$
   then $r > 0$. We claim that $B((x_0,y_0), r) \subset S$. Indeed, let $(x,y) \in B((x_0,y_0), r)$. Then we observe that
$$|x - x_0| = \sqrt{(x-x_0)^2} \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} < r.$$
   Thus we have
$$x = x - x_0 + x_0 \leq |x - x_0| + x_0 < r + x_0 \leq 5 - x_0 + x_0 = 5,$$
   that is, $x < 5$. Similarly,
$$-x = -x + x_0 - x_0 \leq |x_0 - x| - x_0 < r - x_0 \leq x_0 + 5 - x_0 = 5,$$
   that is $-x < 5$ or $x > -5$. Hence $-5 < x < 5$ which implies $(x,y) \in S$. Since $(x,y) \in B((x_0,y_0), r)$ was arbitrary, we have $B((x_0,y_0), r) \subset S$. Since $(x_0,y_0) \in S$ was arbitrary, we have shown that S is open. □

4. (a) **(2 pts)** A set $S \subset E$ is closed if its complement is open. Alternatively, S is closed if whenever a sequence $(x_n)_{n\in\mathbb{N}} \subset S$ converges to some $x \in E$, then $x \in S$.

(b) **(8 pts)** We use the second definition in part (a). Suppose $((x_n, y_n))_{n\in\mathbb{N}} \subset S$ is a sequence converging to $(x, y) \in \mathbb{R}^2$ with respect to the metric d_∞. We will show $(x, y) \in S$. We observe that

$$|x_n - x| \leq \max\{|x_n - x|, |y_n - y|\} = d_\infty((x_n, y_n), (x, y)), \qquad \text{and}$$
$$|y_n - y| \leq \max\{|x_n - x|, |y_n - y|\} = d_\infty((x_n, y_n), (x, y))$$

So if d is the metric on $\mathbb{R}$ from Question 1, then

$$d(x_n, x), d(y_n, y) \leq d_\infty((x_n, y_n), (x, y)).$$

This implies—since $((x_n, y_n))_{n\in\mathbb{N}}$ converges to (x, y) with respect to d_∞—that $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converge to x and y, respectively, with respect to d. Indeed, let $\epsilon > 0$. Then let $N \in \mathbb{N}$ be such that for all $n \geq N$ we have

$$d_\infty((x_n, y_n), (x, y)) < \epsilon.$$

But then by the previous inequalities, for all $n \geq N$ we also have

$$d(x_n, x) < \epsilon \qquad \text{and} \qquad d(y_n, y) < \epsilon.$$

Hence $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. From a proposition in class we therefore obtain:

$$y \cdot x = \lim_{n\to\infty} (y_n \cdot x_n).$$

From yet another proposition from class, since $y_n \cdot x_n \geq 1$ for all $n \in \mathbb{N}$ (by virtue of $(x_n, y_n) \in S$, we have

$$y \cdot x \geq \lim_{n\to\infty} 1 = 1,$$

Hence $(x, y) \in S$. □