Department of Mathematics, University of California, Berkeley

Math 214

Alan Weinstein, Fall 2003

Take Home Final Examination
Due at 825 Evans Hall by 11:15 AM on Friday, December 12th, 2003.
Last update: December 8, 2003, 6:46PM

Instructions. You may use your class notes and the Lee text, but no other references. You should consult nobody except A.W. about the exam. Please send questions about the exam to alanw@math.berkeley.edu **and not to the course emailing list. If I learn of errors or imprecisions on the exam, I will send my corrections to the emailing list (and they will then appear at** http://socrates.berkeley.edu/~alanw/mail-archive.214 **as well).**

Do all of the problems. If you have trouble with one part of a problem, you may still use its result to try the following parts. Unless otherwise specified, all manifolds, maps, flows, actions, ... are C^∞.

1. Let v be a tangent vector at the identity element of a Lie group G, and let v_L and v_R be the left- and right-invariant vector fields on G whose values at the identity element are equal to v. Show that $v_L - v_R$ is a complete vector field, and that this vector field is identically zero if and only if the one parameter subgroup generated by v is contained in the center of G.

2. Let ω be a k-form on the product manifold $M \times Q$, where Q has dimension r and is oriented. We say that ω is **properly supported** if the restriction of the projection $\pi_1 : M \times Q \to M$ to the support of ω is a proper map from the support to M.

(A) Define a linear operation of "integration over the fibres" which assigns to each such ω a $k - r$ form $J(\omega)$ on M such that, if M is oriented and ω has degree $r + \dim M$ and has compact support, then the integral over M of $J(\omega)$ equals the integral over $M \times Q$ of ω.

(B) Show that J can be defined in such a way that, if F is an orientation preserving diffeomorphism of $M \times Q$ such that $\pi_1 \circ F = \pi_1$, then $J(F^*(\omega)) = J(\omega)$.

(C) Now let $\pi : E \to M$ be an orientable (as a bundle) vector bundle of rank r. Show that there is a well-defined operation of integration over the fibres which assigns to each properly supported k form ω on E a $k - r$ form $J(\omega)$ on M such

that, if M is oriented and ω has compact support, then the integral over M of $J(\omega)$ equals the integral over E of ω.

(D) Show that, in the setting of part (C), there is a properly supported r-form λ on E such that $J(\lambda)$ is the constant function 1.

(E) It can be shown that, among such λ's as in part (D), there are closed forms, and all such closed forms lie in the same cohomology class. Show that the class $[\sigma^*\lambda]$ in $H^r_{dR}(M)$ is the same for all sections $\sigma : M \to E$. This class is called the **Euler class** of the oriented vector bundle E.

(F) Show that, if E admits a section which is nowhere zero, then its Euler class is zero.

3. (Be sure to read all parts of this problem before beginning to work on it. You may wish to go right ahead to part (D).) Let δ be a linear operator (i.e. $\mathbb{R}$-linear map) from the space $C^\infty(\mathbb{R})$ of real valued functions on the manifold $\mathbb{R}$ to the space $\Omega^1(\mathbb{R})$ of 1-forms on $\mathbb{R}$. (Be sure to read all parts of this problem before beginning to work on it. You may wish to go right ahead to part (D).)

(A) Find all such operators which commute with Lie derivatives; i.e. $\delta \circ \mathcal{L}_X = \mathcal{L}_X \circ \delta$ for all vector fields X on $\mathbb{R}$.

(B) Find all such operators which commute with the action of diffeomorphisms; i.e. $\delta \circ F^* = F^* \circ \delta$ for all diffeomorphisms $F : \mathbb{R} \to \mathbb{R}$.

(C) Explain (without necessarily proving anything) why the results in parts (A) and (B) are related to one another.

(D) (optional) Do parts (A), (B), and (C) for the more general case where $\mathbb{R}$ is replaced by $\mathbb{R}^n$, or even by an arbitrary manifold. Are the results the same?

4. Let N be a compact submanifold of a manifold M.

(A) Show that there is a smooth function f on M with the following properties: (i) f attains its minimum value of 0 at every point of N, and nowhere else; (ii) at each point of N, the matrix of second derivatives of f in some coordinate system is positive semi-definite, with nullity equal to the dimension of N.

(B) Let f be as in part (A), and let X be a vector field for which $df(X) \le 0$ everywhere on M. Show that there is a neighborhood $\mathcal{U}$ of N such that, for each $p \in \mathcal{U}$, the integral curve for X with initial value p is defined for all *positive* time. Give an example to show that we can not eliminate "positive" from this statement and have it remain true.