UCB Math 121B, Spring 2018: Midterm Exam 2

Prof. Persson, April 5, 2018

Name: Solutions

SID: ____________________

Grading

1	/ 5
2	/ 6
3	/ 5
4	/ 5
	/21

Instructions:

- One sheet of notes, no books, no calculators.
- Exam time 80 minutes, do all of the problems.
- You must justify your answers for full credit.
- Write your answers in the space below each problem.
- If you need more space, use reverse side or scratch pages. Indicate clearly where to find your answers.

1. (5 points) Find *all solutions* to the following differential equation by the method of Frobenius (generalized power series).

$$4xy'' + (x-1)y' + y = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+s}$$

$$-y' = -\sum_{n=0}^{\infty} (n+s) a_n x^{n+s-1}$$

$$xy' = \sum_{n=0}^{\infty} (n+s) a_n x^{n+s}$$

$$4xy'' = 4\sum_{n=0}^{\infty} (n+s)(n+s-1) a_n x^{n+s-1}$$

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x^{s-1}: $\quad 4s(s-1)a_0 - s a_0 = 0$

$\quad s(4s-5)a_0 = 0 \quad \Rightarrow \quad \begin{cases} s = 0 \\ s = 5/4 \end{cases}$

$s=0$: $\quad x^s$: $\quad 0a_1 + 0a_0 - 1a_1 + a_0 = 0 \quad \Rightarrow \quad a_1 = a_0$

$x^{n+s}, n>0$: $\quad [4n(n+1) - (n+1)]\, a_{n+1} + (n+1)a_n = 0$

$\Rightarrow a_{n+1} = -\frac{1}{4n-1} a_n \quad \Rightarrow \quad y = a_0\left(1 + x - \frac{x^2}{3} + \frac{x^3}{7\cdot 3} - \cdots\right)$

$s = 5/4$: $\quad x^{n+s}, n \geq 0$: $\quad [4(n+s)(n+s-1) - (n+s+1)]\, a_{n+1} + (n+s+1)a_n = 0$

$\Rightarrow a_{n+1} = -\frac{1}{4(n+1)} a_n \quad \Rightarrow \quad y = a_0 x^{5/4}\left(1 - \frac{x}{4} + \frac{x^2}{8\cdot 4} - \frac{x^3}{12\cdot 8\cdot 4} + \cdots\right)$

2. Recall the definition of the Bessel function of the first kind of order p:

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \left(\frac{x}{2}\right)^{2n+p}$$

a) (3 points) Show that $\frac{d}{dx}\left[x^{-p}J_p(x)\right] = -x^{-p}J_{p+1}(x)$.

$$LHS = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+1+p)} \cdot \frac{x^{2n}}{2^{2n+p}}\right] = \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 2n}{\Gamma(n+1)\Gamma(n+1+p)} \cdot \frac{x^{2n-1}}{2^{2n+p}}$$

$$= [n \to n+1] = -\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2\cdot(n+1)\cdot x^{2n+1}}{\Gamma(n+2)\Gamma(n+2+p)\cdot 2^{2n+2+p}}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{\Gamma(n+1)\Gamma(n+1+p+1)\, 2^{2n+p+1}} = RHS$$

b) (3 points) Evaluate the integral $\int_0^\infty x^{-p}J_{p+1}(x)\,dx$.

$$-\int_0^\infty \frac{d}{dx}\left[x^{-p}J_p(x)\right]dx = -\lim_{x\to\infty} x^{-p}J_p(x) + \lim_{x\to 0} x^{-p}J_p(x)$$

$$= 0 + \frac{(-1)^0}{\Gamma(0+1)\Gamma(0+1+p)} \cdot \frac{1}{2^{2\cdot 0+p}} = \frac{1}{2^p\,\Gamma(p+1)}$$

3. Find the solutions of the following differential equations, using the fact that

$$y'' + \frac{1-2a}{x}y' + \left[(bcx^{c-1})^2 + \frac{a^2 - p^2c^2}{x^2}\right] y = 0 \quad \Longrightarrow \quad y = x^a Z_p(bx^c)$$

a) (2 points) $x^2y'' - 2xy' + (1+x^2)y = 0$

$$y'' - \frac{2}{x}y' + \left(1 + \frac{1}{x^2}\right)y = 0$$

$$\begin{cases} 1-2a=-2 \\ c-1=0 \\ b^2c^2=1 \\ a^2-p^2c^2=1 \end{cases} \Rightarrow \begin{cases} a=3/2 \\ c=1 \\ b=1 \\ \frac{9}{4}-p^2=1, \; p=\frac{\sqrt{5}}{2} \end{cases} \Rightarrow y = x^{3/2} Z_{\sqrt{5}/2}(x)$$

b) (3 points) $y'' + 2y' + (e^{4x} - 1)y = 0$
Hint: Change independent variable to $z = e^x$, then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = z\frac{dy}{dz}$, etc.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(z\frac{dy}{dz}\right) = [\text{Product rule}] = \frac{dz}{dx}\frac{dy}{dz} + z\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= z\frac{dy}{dz} + z\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = z\frac{dy}{dz} + z^2\frac{d^2y}{dz^2}$$

$$\Rightarrow \quad z^2y'' + zy' + 2zy' + (z^4-1)y = 0$$

$$y'' + \frac{3}{z}y' + \left(z^2 - \frac{1}{z^2}\right)y = 0$$

$$\begin{cases} 1-2a=3 \\ 2c-2=2 \\ b^2c^2=1 \\ a^2-p^2c^2=-1 \end{cases} \Rightarrow \begin{cases} a=-1 \\ c=2 \\ b=1/2 \\ 1-p^2 4=-1, \\ p=1/\sqrt{2} \end{cases} \Rightarrow y = z^{-1}Z_{1/\sqrt{2}}\left(\frac{z^2}{2}\right)$$

$$= e^{-x} Z_{1/\sqrt{2}}\left(\frac{e^{2x}}{2}\right)$$

4. Consider the one-dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$.

a) (2 points) Use separation of variables to show that the solutions have the form

$$y = \begin{Bmatrix} \sin kx \\ \cos kx \end{Bmatrix} \begin{Bmatrix} \sin kvt \\ \cos kvt \end{Bmatrix}, \qquad k \geq 0$$

$y = X(x)T(t)$

$\frac{1}{X} X''(x) = \frac{1}{v^2 T} T''(t) = -k^2, \quad k \geq 0$

$X''(x) = -k^2 X(x) \Rightarrow X = \begin{cases} \sin kx \\ \cos kx \end{cases}$

$T''(t) = -k^2 v^2 T(t) \Rightarrow T = \begin{cases} \sin kvt \\ \cos kvt \end{cases}$

b) (3 points) Use this to find the displacement $y(x,t)$ of a string with wave speed $v = 1$, whose left end $x = 0$ is free ($\partial y/\partial x = 0$) and right end $x = \pi$ is pinned ($y = 0$), if at time $t = 0$ the initial displacement is $y = \cos(3x/2)$ and the initial velocity is $\partial y/\partial t = -\cos(x/2)$.

- $\frac{\partial y}{\partial x} = 0$ at $x = 0 \Rightarrow$ discard $\sin kx$
- $y = 0$ at $x = \pi \Rightarrow \cos(k\pi) = 0 \Rightarrow k = n + 1/2$
- Match initial displacement: $\cos\frac{3x}{2}\cos\frac{3t}{2}$
- Match initial velocity: $-\cos\frac{x}{2}\cdot\sin\frac{t}{2}\cdot\left(\frac{1}{1/2}\right)$

$\Rightarrow y(x,t) = \cos\frac{3x}{2}\cos\frac{3t}{2} - 2\cos\frac{x}{2}\sin\frac{t}{2}$