Midterm #2, Physics 5C, Spring 2018. Write your responses below, on the back, or on the extra pages. Show your work, and take care to explain what you are doing; partial credit will be given for incomplete answers that demonstrate some conceptual understanding. Cross out or erase parts of the problem you wish the grader to ignore. **Some potentially useful formulae are given on the back page**

Problem 1: (16 pts)

A particle in 3D space has a wavefunction[1] in spherical coordinates

$$\psi(r,\theta,\phi) = \frac{A}{r}e^{-\alpha r}e^{2i\phi} \tag{1}$$

where A and α are real constants.

1a) Determine A

Solution: We must normalize the solution, which in spherical coordinates means

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \psi^*\psi r^2 dr \sin\theta d\theta d\phi = 1 \tag{2}$$

In this case

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{A}{r}e^{-\alpha r}e^{-2i\phi}\frac{A}{r}e^{-\alpha r}e^{2i\phi}r^2 dr \sin\theta d\theta d\phi = 1 \tag{3}$$

$$A^2\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-2\alpha r} dr \sin\theta d\theta d\phi = 1 \tag{4}$$

Since there is no dependence on θ, ϕ, the angular integrals just give a factor of 4π

$$4\pi A^2 \int_0^{\infty} e^{-2\alpha r} dr = 1 \tag{5}$$

Doing the radial integral

$$4\pi A^2 \frac{1}{-2\alpha}e^{-2\alpha r}\Big|_0^{\infty} = \frac{2\pi A^2}{\alpha} = 1 \tag{6}$$

From which we find

$$A = \sqrt{\frac{\alpha}{2\pi}} \tag{7}$$

So that the normalized wavefunction is

$$\psi = \sqrt{\frac{\alpha}{2\pi}}\frac{e^{-\alpha r}e^{2i\phi}}{r} \tag{8}$$

1b) At what radius r_0 is the probability of finding the particle at $r > r_0$ equal to 1/2? Give the result in terms of the constants given.

Solution: The probability that we find the particle at $r > r_0$ is

$$P(r > r_0) = \int_0^{2\pi}\int_0^{\pi}\int_0^{r_0} |\psi|^2 r^2 dr \sin\theta d\theta d\phi \tag{9}$$

The angular integrals just give 4π again so

$$P(r > r_0) = 4\pi\int_{r_0}^{\infty} \frac{\alpha}{2\pi}e^{-2\alpha r}dr = 2\alpha\left(\frac{1}{-2\alpha}\right)e^{-2\alpha r}\Big|_{r_0}^{\infty} = e^{-2\alpha r_0} \tag{10}$$

We want probability one half so

$$e^{-2\alpha r_0} = 1/2 \quad \Rightarrow e^{2\alpha r_0} = 2 \quad \Rightarrow \quad 2\alpha r_0 = \log 2 \tag{11}$$

[1] Don't worry about the fact that the wavefunction is infinite at $r = 0$; that won't affect the results of this problem. If you like you can imagine that ψ saturates to some finite value for very small r.

Thus

$$r_0 = \frac{\log 2}{2}\frac{1}{\alpha} \tag{12}$$

1c) The operator associated with orbital angular momentum in the z direction is

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi} \tag{13}$$

Show that the wavefunction ψ an eigenstate of $\hat{L}_z$.

Solution: We apply $\hat{L}_z$ to the function

$$\hat{L}_z\psi = -i\hbar\frac{\partial\psi}{\partial\phi} = -i\hbar\frac{\partial}{\partial\phi}\left(\frac{A}{r}e^{-\alpha r}e^{i2\phi}\right) = -i\hbar\left(\frac{A}{r}e^{-\alpha r}\frac{\partial e^{i2\phi}}{\partial\phi}\right) = -i\hbar\left(i2\frac{A}{r}e^{-\alpha r}e^{i2\phi}\right) \tag{14}$$

And so

$$\hat{L}_z\psi = -i\hbar(i2)\psi = 2\hbar\psi \tag{15}$$

So $\hat{L}_z$ applied to ψ returns ψ times a constant. The eigenvalue is $2\hbar$.

1d) What is the expectation value of $\hat{L}_z$ for this ψ? What is the uncertainty in $\hat{L}_z$?

Since ψ is an eigenstate, we know that a measurement will return the eigenvalue $2\hbar$ every time. So we can say off the bat that $\langle L_z\rangle = 2\hbar$ and $\sigma_{Lz} = 0$. If we want to show this explicitly (not necessary to do, but a reasonable double check) we can do the integrals

$$\langle L_z\rangle = \int\int\int(\psi^*\hat{L}_z\psi)r^2 dr\sin\theta d\theta d\phi = \int\int\int(\psi^* 2\hbar\psi)r^2 dr\sin\theta d\theta d\phi \tag{16}$$

$$= 2\hbar\int\int\int|\psi|^2 r^2 dr\sin\theta d\theta d\phi = 2\hbar \tag{17}$$

where in the last step we used the fact that we already normalized ψ above. We also have similarly

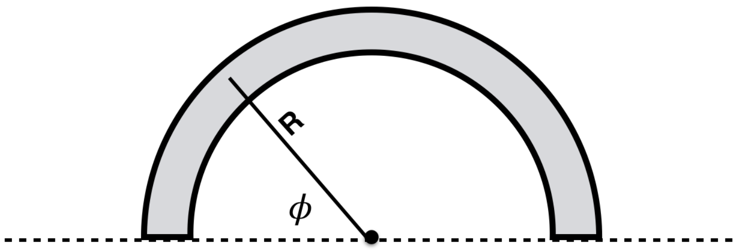
$$\langle L_z^2\rangle = \int\int\int(\psi^*\hat{L}_z^2\psi)r^2 dr\sin\theta d\theta d\phi = \int\int\int(\psi^* 2\hbar\hat{L}_z\psi)r^2 dr\sin\theta d\theta d\phi \tag{18}$$

$$== \int\int\int(\psi^*(2\hbar)^2\psi)r^2 dr\sin\theta d\theta d\phi = 4\hbar^2\int\int\int|\psi|^2 r^2 dr\sin\theta d\theta d\phi = 4\hbar^2 \tag{19}$$

and so

$$\sigma_{Lz}^2 = \langle L_z^2\rangle - \langle L_z\rangle^2 = 4\hbar^2 - (2\hbar)^2 = 0 \tag{20}$$

As expected



Problem 2: (20 pts) A particle is in a semi-circular infinite well where $V = 0$ inside the well (grey shaded region in the figure) and $V = \infty$ outside. In spherical coordinates, the well confines the particle at a constant radius $r = R$ and constant polar angle $\theta = \pi/2$. The wavefunction thus only depends on the angle ϕ, i.e., $\psi = \psi(\phi)$

2a) Solve for the normalized energy eigenstates of this particle.

Solution: The time-independent Schrodinger equation is

$$\frac{-\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi \tag{21}$$

Since the wavefunction is independent of θ, r we can drop the derivatives with respect to those functions in the Laplacian. The equation then becomes

$$\frac{-\hbar^2}{2mR^2}\frac{\partial^2\psi}{\partial\psi^2} = E\psi \tag{22}$$

Rewriting this as

$$\frac{\partial^2\psi}{\partial\psi^2} = -k^2\psi \quad \text{where } k = \sqrt{\frac{2mER^2}{\hbar^2}} \tag{23}$$

The solution is

$$\psi = A\sin(k\phi) + B\cos(k\phi) \tag{24}$$

Or alternatively we could write it in terms of complex exponentials

$$\psi = A'e^{ik\phi} + B'e^{-ik\phi} \tag{25}$$

The boundary conditions give that $\psi = 0$ at $\phi = 0$ and $\phi = \pi$. These first gives

$$\psi(0) = B\cos(k\pi/2) = 0 \qquad \Rightarrow \quad B = 0 \tag{26}$$

The second gives

$$\psi(\pi) = A\sin(k\pi) = 0 \qquad \Rightarrow \quad k = n \;\text{ where } n = 1, 2, 3, ... \tag{27}$$

So the solutions are

$$\psi_n(\phi) = A\sin(n\phi) \tag{28}$$

To normalize we want

$$\int_0^\pi |A|^2\sin^2(n\phi)d\phi = 1 \tag{29}$$

This looks a lot like a particle in a 1D well. If we wanted to make it look more familiar for doing the integral, we could make the substitution $\phi = \pi x$

$$\int_0^1 |A|^2\sin^2(n\pi x)\pi dx = 1 \tag{30}$$

Apart from the extra factor of π, this is the same normalization integral we do for the 1D particle in a box of length $L = 1$. We know that the integration of the $\sin^2$ over a box gives a factor of $1/2$ so

$$\int_0^1 |A|^2 \sin^2(n\pi x)\pi dx = |A|^2 \frac{\pi}{2} = 1 \tag{31}$$

And so $|A|^2 = 2/\pi$ or $A = \sqrt{2/\pi}$ up to an arbitrary complex phase. The normalization is the same as a box of length π. The normalized wavefunction

$$\psi = \sqrt{\frac{2}{\pi}} \sin(n\phi) \tag{32}$$

In the above, we considered ψ a function of only ϕ, and so $|\psi|^2$ has units of per radian.

2b) Determine the values of energy that could be measured for the particle.

Solution: From above, the energy is given by

$$E = \frac{\hbar^2 k^2}{2mR^2} \tag{33}$$

And with the quantization condition $k = n$ the allowed energies are

$$E_n = \frac{\hbar^2 n^2}{2mR^2} \qquad n = 1, 2, 3, ... \tag{34}$$

Problem 3: (20 pts)

Consider a particle of spin 1/2. We will use the eigenstates of the $\hat{S}_z$ operator as the basis vectors, such that

$$\chi_{z,\uparrow} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi_{z,\downarrow} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{35}$$

where $\chi_{z,\uparrow}$ represents spin-up in the z direction, and $\chi_{z,\downarrow}$ spin down in z.

The particle if is put into a uniform magnetic field pointing perpendicular to the z direction, such that the energy associated with the particle's spin is given by the Hamiltonian

$$\hat{H} = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix} \tag{36}$$

where ϵ is a constant (units of energy).

3a) Calculate the possible values of energy that could be measured for this system.

Solution: We want to find the eigenvalues of the matrix. We use the standard approach

$$det|\hat{H} - \lambda \hat{I}| = \begin{vmatrix} -\lambda & \epsilon \\ \epsilon & -\lambda \end{vmatrix} = \epsilon^2 - \lambda^2 = 0 \qquad \Rightarrow \lambda = \pm\epsilon \tag{37}$$

3b) Find the normalized eigenstates of the Hamiltonian

Solution We find the states

$$\hat{H} \begin{pmatrix} a \\ b \end{pmatrix} = \epsilon \begin{pmatrix} b \\ a \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \tag{38}$$

The first component of this equation gives

$$\epsilon b = \pm \epsilon a \qquad \Rightarrow b = \pm a \tag{39}$$

The energy eigenstates are thus

$$\vec{v_1} = \begin{pmatrix} a \\ a \end{pmatrix} \qquad \vec{v_2} == \begin{pmatrix} a \\ -a \end{pmatrix} \tag{40}$$

Normalizing these vectors such that $a^2 + a^2 = 1$ implies $a = 1/\sqrt{2}$ and so

$$\vec{v_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{v_2} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{41}$$

3c) We measure the particle's energy and find it to be in the lowest possible energy state. What is the probability that a measurement of $\hat{S}_z$ returns $+\hbar/2$ (i.e., spin up in the z direction)?

Solution The measurement collapses the particle to the $\vec{v}_2$ state. This is a superposition of $\hat{S}_z$ eigenstates, as we can see

$$\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \vec{\chi}_{z,\uparrow} - \frac{1}{\sqrt{2}} \vec{\chi}_{z,\downarrow} \tag{42}$$

By the quantum postulates, the probability is the coefficient in front of $\vec{\chi}_{z,\uparrow}$ squared, and so it is $P = 1/2$.

3d) A particle is initially the $\chi_{z,\uparrow}$ state. We wait some time t and measure the z-component of spin. What is the probability that the measurement finds the particle to be in the $\chi_{z,\downarrow}$ state?

Solution: The problem is essentially similar to the neutrino oscillation (or rabbit-duck) problem done on the homework. The eigenstates of the Hamiltonian are the stationary states. The initial state given can be written as a superposition of energy eigenstates

$$\chi_{z,\uparrow} = \frac{1}{\sqrt{2}} \vec{v}_1 + \frac{1}{\sqrt{2}} \vec{v}_2 \tag{43}$$

The time-dependence of the energy eigenstates are just given by a phase factor $e^{-i\omega t}$. Define $\omega_1 = E_1/\hbar$ and $\omega_2 = E_2/\hbar$. The time-dependent result is

$$\chi(t) = \frac{1}{\sqrt{2}} e^{-\omega_1 t} \vec{v}_1 + \frac{1}{\sqrt{2}} e^{-\omega_2 t} \vec{v}_2 \tag{44}$$

Plugging in our results for $\vec{v}_1, \vec{v}_2$

$$\chi(t) = \frac{1}{\sqrt{2}} e^{-\omega_1 t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} e^{-\omega_2 t} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tag{45}$$

$$= \frac{1}{2} \begin{pmatrix} e^{-i\omega_1 t} + e^{-i\omega_2 t} \\ e^{-i\omega_1 t} - e^{-i\omega_2 t} \end{pmatrix} \tag{46}$$

The probability we want is the second component squared (corresponding to spin down in z). This is

$$P(z_\downarrow) = \frac{1}{\sqrt{2}} \left(e^{i\omega_1 t} - e^{i\omega_2 t} \right) \frac{1}{\sqrt{2}} \left(e^{-i\omega_1 t} - e^{-i\omega_2 t} \right) \tag{47}$$

$$= \frac{1}{2} \left(1 - e^{i\omega_1 t - i\omega_2 t} - e^{i\omega_2 t - i\omega_1 t} + 1 \right) \tag{48}$$

$$= \frac{1}{2} \left(2 - 2\cos((\omega_2 - \omega_1)t) \right) \tag{49}$$

Here $\omega_2 - \omega_1 = \epsilon/\hbar - (-\epsilon/\hbar) = 2\epsilon/\hbar$ and so

$$P(z_\downarrow) = 1 - \cos(2\epsilon t/\hbar) \tag{50}$$

We check that the probability is bounded by 0 and 1. If we wanted to double-check, we could also calculate that

$$P(z_\uparrow) = 1 + \cos(2\epsilon t/\hbar) \tag{51}$$

So that $P(z_\uparrow) + P(z_\downarrow) = 1$.

Another manipulation of the algebra would be to write the bottom componet of the array as

$$c_\downarrow = \frac{1}{2}(e^{-i\omega_1 t} - e^{-i\omega_2 t}) = \frac{1}{2} e^{-i\omega_1 t/2} e^{-i\omega_2 t/2} (e^{i\Delta\omega t} - e^{-i\Delta\omega t}) \tag{52}$$

where $\Delta\omega = (\omega_2 - \omega_1)/2 = \epsilon/\hbar$. This can then be written

$$c_\downarrow = \frac{1}{2} e^{-i\omega_1 t/2} e^{-i\omega_2 t/2} 2\sin(\Delta\omega t) = e^{-i(\omega_1 + \omega_2)t/2} \sin(\Delta\omega t) \tag{53}$$

Then the probability is

$$P(z_\downarrow) = |c_\downarrow|^2 = \sin^2(\epsilon t/\hbar) \tag{54}$$

This is equivalent to the above result (as can be shown from using trig identities)

Problem 4: (20 pts)

A certain thermodynamic system of fixed volume is described by the macroscopic variables U (internal energy) and N (total number of particles). The number of microstates corresponding to a macrostate U, N turns out to be given by

$$\Omega(U,N) = \left[C(U/N)^{3/2}\right]^N \tag{55}$$

where C is a constant.

4a) Determine an expression for the system temperature as a function of U and N.

Solution: We first calculate the entropy

$$S = Nk_B \log\Omega = Nk_B \log\left(C(U/N)^{3/2}\right) = Nk_B \log C + Nk_B \log\left(U^{3/2}\right) - Nk_B \log\left(N^{3/2}\right) \tag{56}$$

Using additional properties of the log this becomes

$$S = Nk_B \log C + \frac{3}{2}Nk_B \log U - \frac{3}{2}k_B \log N \tag{57}$$

The temperature is defined as

$$\frac{1}{T} = \left.\frac{\partial S}{\partial U}\right|_N = \frac{3}{2}\frac{Nk_B}{U} \tag{58}$$

and so

$$T = \frac{2}{3}\frac{U}{Nk_B} \tag{59}$$

Now consider two such systems. System 1 is initially in a macrostate ($U_1 = 1$ J, $N_1 = 1\ N_A$) while System 2 is in a macrostate ($U_2 = 2\ JN_2 = 10N_A$). Here J denotes the energy unit Joules and N_A is Avogadro's number.

4b) The two systems are put into thermal contact (the number of particles in each system is held fixed). Is energy most likely to flow from System 1 to System 2 or vice versa?

Solution The temperatures are

$$T_1 = \frac{2}{3}\frac{1J}{N_Ak_B} = \frac{2}{3}\frac{J}{N_Ak_B} \qquad T_2 = \frac{2}{3}\frac{2J}{10N_Ak_B} = \frac{4}{30}\frac{J}{N_Ak_B} = \frac{2}{15}\frac{J}{N_Ak_B} \tag{60}$$

we see that $T_2 < T_1$ so energy is likely to flow from System 1 to System 2.

4c) After these two systems come into thermal equilibrium, what is the energy of System 1 in Joules?

Solution The temperatures will be equal in equilibrium, thus

$$T_1 = T_2 \quad \Rightarrow \frac{2}{3}\frac{U_1'}{N_1k_B} = \frac{2}{3}\frac{U_2'}{N_2k_B} \tag{61}$$

where we use U_1', U_2' to represent the final energies of the systems. This gives

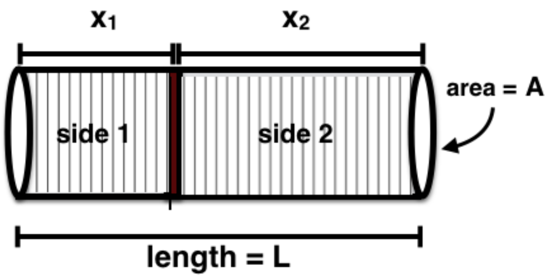
$$\frac{U_1'}{N_1} = \frac{U_2'}{N_2} \quad \Rightarrow\ U_1' = U_2'\frac{N_1}{N_2} = U_2'\frac{1}{10} \tag{62}$$

The total energy is a constant $U = U_1 + U_2 = 3\ J$ and so $U_2' = U - U'1$, which gives

$$U_1' = (U - U_1')\frac{1}{10} \quad \Rightarrow\ U_1'\left(1 + \frac{1}{10}\right) = U\frac{1}{10} \tag{63}$$

$$U_1'(10+1) = U \quad \Rightarrow 11U_1' = U \tag{64}$$

And since $U = 3J$ we have $U_1' = 3/11\ J$. System 1 did indeed gain energy.



Problem 5: (10 pts)

A tube of length L is filled with gas, which is divided into two sides by a moveable wall. Initially, side 1 has volume $V_1 = x_1 A$ and side 2 has volume $V_2 = x_2 A$. We fix the number of particles in each side to be $N_1 = 2N_A$ and $N_2 = N_A$. Both sides of the tube are kept at a constant temperature T.

5a) If the gas is an ideal gas, find the position x_1 (in terms of L) of the wall where the pressures on both sides of the wall are equal, $P_1 = P_2$.

Solution: The ideal gas law is

$$P = \frac{N k_B T}{V} \tag{65}$$

The balance $P_1 = P_2$ then implies

$$\frac{N_1 k_B T}{V_1} = \frac{N_2 k_B T}{V_2} \quad \Rightarrow \frac{N_1}{V_1} = \frac{N_2}{V_2} \tag{66}$$

using $V = xA$ and $N_1 = 2N_2$

$$\frac{N_1}{x_1} = \frac{2N_1}{x_2} \quad \Rightarrow \frac{1}{x_1} = \frac{2}{x_2} \quad \Rightarrow x_2 = 2x_1 \tag{67}$$

Since $x_1 + x_2 = L$ we have $x_2 = L - x_1$

$$x_2 = 2(L - x_1) \quad \Rightarrow 3x_1 = 2L \quad \rightarrow x_1 = \frac{2}{3}L \tag{68}$$

Now consider the problem statistically. Imagine slicing up the tube into many small elements of length Δx such that there are $M_1 = x_1/\Delta x$ slices on Side 1 and $M_2 = x_2/\Delta x$ slices on Side 2. The number of ways of distributing N particles among M elements is[2]

$$\Omega = \frac{(N + M - 1)!}{(M - 1)! N!} \tag{69}$$

We assume that $N \gg 1$ and $M \gg 1$.

5b) Assuming the wall can move freely and that the ergodic hypothesis holds, calculate the location x_1 (in terms of L) of the wall once equilibrium is reached.

Solution The entropy of the combined system is

$$S = S_1 + S_2 = k_B \log \Omega_1 + k_B \log \Omega_2 \tag{70}$$

We want to maximize this with respect to the variable x_1, which gives

$$\frac{\partial S}{\partial x_1} = \frac{\partial S_1}{\partial x_1} + \frac{\partial S_2}{\partial x_1} = 0 \tag{71}$$

[2]We are ignoring any additional degrees of freedom that may add to the number of microstates due to e.g., the energy of the particles.

Now $x_2 = L - x_1$ so $dx_2 = -dx_1$. Using the chain rule

$$\frac{\partial S}{\partial x_1} = \frac{\partial S_1}{\partial x_1} + \left(\frac{\partial S_2}{\partial x_2}\right)\left(\frac{\partial x_2}{\partial x_1}\right) = \frac{\partial S_1}{\partial x_1} - \frac{\partial S_2}{\partial x_2} = 0 \tag{72}$$

So the maximize system entropy occurs when

$$\frac{\partial S_1}{\partial x_1} = \frac{\partial S_2}{\partial x_2} \tag{73}$$

This is analogous to the equation when we put systems in thermal equilibrium. In fact, this derivative of S is related to the statistical definition of pressure.

The problem gives $x_1 = M_1 \Delta x$ and $x_2 = M_2 \Delta x$. So equilibrium can be written

$$\frac{\partial S_1}{\partial M_1} = \frac{\partial S_2}{\partial M_2} \tag{74}$$

From the given Ω, we can calculate the entropy of a system

$$S = k_B \log \Omega = k_B \log[(N + M - 1)!] - k_B \log[(M-1)!] - k_b \log[N!] \tag{75}$$

Using Sterling's approximation

$$\frac{S}{k_B} = (N+M-1)\log(N+M-1) - (N+M-1) - (M-1)\log(M-1) + (M-1) - NlogN + N \tag{76}$$

$$= (N+M-1)\log(N+M-1) - (M-1)\log(M-1) - N\log N \tag{77}$$

From this we get

$$\frac{1}{k_B}\frac{\partial S}{\partial M} = \log(N+M-1) + 1 - \log(M-1) - 1 = \log(N+M-1) - \log(M-1) \tag{78}$$

and so

$$\frac{\partial S}{\partial M} = k_B \log\left[\frac{N+M-1}{M-1}\right] \tag{79}$$

Setting this quantity equal on both sides gives

$$k_B \log\left[\frac{N_1+M_1-1}{M_1-1}\right] = k_B \log\left[\frac{N_2+M_2-1}{M_2-1}\right] \tag{80}$$

exponentiating

$$\frac{N_1+M_1-1}{M_1-1} = \frac{N_2+M_2-1}{M_2-1} \tag{81}$$

$$\frac{N_1}{M_1-1} - 1 = \frac{N_2}{M_2-1} - 1 \tag{82}$$

$$\frac{N_1}{M_1-1} = \frac{N_2}{M_2-1} \tag{83}$$

using the now that $N_1 = 2N_2$

$$\frac{2N_1}{M_1-1} = \frac{N_1}{M_2-1} \quad \Rightarrow \frac{2}{M_1-1} = \frac{1}{M_2-1} \tag{84}$$

Since $M \gg 1$ we can drop the 1's (we could have done this much earlier...)

$$M_1 = 2M_2 \tag{85}$$

Multiply both sides by Δx

$$\Delta x M_1 = 2\Delta x M_2 \quad \Rightarrow x_1 = 2x_2 \quad \Rightarrow x_1 = 2(L - x_1) \tag{86}$$

From which we find

$$x_1 = \frac{2}{3}L \tag{87}$$

As found with the ideal gas law, but from a statistical calculation.