Math H54 **Midterm Exam #2** November 9, 2017

Complete all problems. You must show your work or justify your answer for all problems. *Answers without work or justification will receive no credit (even if they are correct).* You may quote theorems and results from class or the homework without justification (name the theorem or state "we proved in class that ..."). Every other fact must be justified. If you need more space, use the blank pages at the back of the exam. If you want me to grade work done any of those pages, clearly indicate this next to the appropriate problem.

Problem 1		6
Problem 2		6
Problem 3		6
Problem 4		6
Problem 5		15
Problem 6		15
Problem 7		14
Problem 8		9
Total		77

Name SOLUTIONS

1. *(6 points)* Show that the only real matrix that is orthogonal, symmetric, and has all positive eigenvalues is the identity matrix.

A orth $\Rightarrow A^{-1} = A^T$. A symm $\Rightarrow A = A^T$

So $A^2 = AA^T = I$. A symm $\Rightarrow$ orth. diag $\Rightarrow A = P^T D P$.

$I = A^2 = P^T D^2 P \Rightarrow D^2 = I \Rightarrow$ entries are $\pm 1 \Rightarrow$

entries are 1, since A has all positive e-val's.

$\Rightarrow D = I$.

Thus, $A = P^T I P = P^T P = I$.

2. *(6 points)* Let A be a real $n \times n$ matrix. Prove that if $\|A\vec{x}\| = \|\vec{x}\|$ for every $\vec{x} \in \mathbb{R}^n$, then

$$\langle A\vec{x}, A\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$$

for every $\vec{x}, \vec{y} \in \mathbb{R}^n$.

$\langle A(\vec{x}+\vec{y}), A(\vec{x}+\vec{y})\rangle = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle$

$\langle A\vec{x}, A\vec{x}\rangle + 2\langle A\vec{x}, A\vec{y}\rangle + \langle A\vec{y}, A\vec{y}\rangle = \langle \vec{x}, \vec{x}\rangle + 2\langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{y}\rangle$

$\cancel{\langle \vec{x}, \vec{x}\rangle} + 2\langle A\vec{x}, A\vec{y}\rangle + \cancel{\langle \vec{y}, \vec{y}\rangle} = \cancel{\langle \vec{x}, \vec{x}\rangle} + 2\langle \vec{x}, \vec{y}\rangle + \cancel{\langle \vec{y}, \vec{y}\rangle}$

$\Rightarrow \langle A\vec{x}, A\vec{y}\rangle = \langle \vec{x}, \vec{y}\rangle$ ✓

3. *(6 points)* Find the homogeneous linear differential equation equation with constant coefficients *of least order* that has

$$y = 2e^{3t} + 3\sin(2t) + 2t$$

as a solution.

$r=3,\ r=\pm 2i,\ r=0$ order 2

$r^2(r-3)(r^2+4) = r^2(r^3-3r^2+4r-12) = r^5-3r^4+4r^3-12r^2$

$$\boxed{y^{(5)} - 3y^{(4)} + 4y''' - 12y'' = 0.}$$

4. *(3 points each)* Assume that $y_1(t) = \cos t$ and $y_2(t) = t$ are both solutions of the equation $ay'' + by' + cy = q(t)$ for a certain function $q(t)$.

(a) Find a nonzero solution of the homogeneous equation $ay'' + by' + cy = 0$.

$\boxed{y(t) = \cos t - t}$ $\quad (q(t) - q(t) = 0)$

* Alternatively, you could show that there is no such differential equation unless $a=b=c=0$.

(b) Find a solution $y(t)$ of $ay'' + by' + cy = q(t)$ such that $y(0) = 2$.

$y_1(0) = 1 \quad y_2(0) = 0.$

$\boxed{y(t) = 2\cos t - t}$ $\quad (2q(t) - q(t) = q(t))$

5. Consider the inner product space $\mathcal{C}([-1,1])$ with the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx.$$

(a) *(10 points)* Construct an orthonormal basis (with respect to this inner product) for the subspace $\mathbb{P}_2$ of polynomials with real coefficients of degree at most two.

$f_1 = 1.\quad \|f_1\|^2 = \int_{-1}^{1} 1\,dx = 2 \longrightarrow 1/\sqrt{2} = \hat{f}_1$

$f_2 = x$, b/c $\int_{-1}^{1} x\,dx = 0.\quad \|x\|^2 = \int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}$

$\rightarrow \sqrt{\frac{3}{2}}\,x = \hat{f}_2$

$f_3 = x^2 - \langle x^2, 1/\sqrt{2}\rangle \cdot 1/\sqrt{2} - \langle x^2, \sqrt{\tfrac{3}{2}}x\rangle \cdot \sqrt{\tfrac{3}{2}}x$

- $\frac{1}{\sqrt{2}}\langle x^2, 1/\sqrt{2}\rangle = \frac{1}{2}\int_{-1}^{1} x^2\,dx = \frac{1}{2}\cdot\frac{2}{3} = 1/3$
- $\langle x^2, \sqrt{\tfrac{3}{2}}x\rangle = 0$

$\Rightarrow f_3 = x^2 - \frac{1}{3}.$

$\|x^2 - 1/3\|^2 = \int_{-1}^{1}(x^2 - 1/3)^2\,dx = \int_{-1}^{1} x^4 - \frac{2}{3}x^2 + 1/9\,dx = \frac{8}{45}$

$\rightarrow \sqrt{\frac{45}{8}}(x^2 - 1/3) = \hat{f}_3$

$$\boxed{\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\,x, \sqrt{\frac{45}{8}}(x^2 - 1/3)\right\}}$$

(b) *(5 points)* Find the best approximation (with respect to this inner product) to $f(x) = x^5$ by polynomials in $\mathbb{P}_2$.

$$\left\langle x^5, \frac{1}{\sqrt{2}} \right\rangle = 0 \ , \ \left\langle x^5, \sqrt{\frac{45}{8}}\left(x^2 - 1/3\right) \right\rangle = 0$$

$$\left\langle x^5, \sqrt{\frac{3}{2}}\, x \right\rangle = \sqrt{\frac{3}{2}} \int_{-1}^{1} x^6 \, dx = \sqrt{\frac{3}{2}} \cdot \frac{2}{7}$$

$$x^5 \approx \frac{3}{2} \cdot \frac{2}{7} x = \boxed{\frac{3}{7} x}$$

6. Consider the differential equation

$$y'' - 4y' - 12y = 80\sin(2t).$$

(a) *(4 points)* Find the general solution to the corresponding homogeneous equation $y'' - 4y' - 12y = 0$.

$$r^2 - 4r - 12 = 0$$

$$(r-6)(r+2) = 0$$

$$r = 6, \quad r = -2$$

$$\boxed{y(t) = C_1 e^{6t} + C_2 e^{-2t}}$$

(b) *(5 points)* Find a particular solution to the non-homogeneous equation $y'' - 4y' - 12y = 80\sin(2t)$.

$$y_p(t) = A\cos 2t + B\sin 2t$$
$$y_p'(t) = -2A\sin 2t + 2B\cos 2t$$
$$y_p''(t) = -4A\cos 2t - 4B\sin 2t$$

$$(-4A\cos 2t - 4B\sin 2t) - 4(-2A\sin 2t + 2B\cos 2t) - 12(A\cos 2t + B\sin 2t)$$

$$(-4A - 8B - 12A)\cos 2t + (-4B + 8A - 12B)\sin 2t = 80\sin 2t$$

$$-16A - 8B = 0 \rightarrow -2A = B$$

$$8A - 16B = 80 \rightarrow A - 2B = 10 \rightarrow A - 2(-2A) = 10$$

$$5A = 10 \rightarrow A = 2 \Rightarrow B = -4$$

$$\boxed{y_p(t) = 2\cos(2t) - 4\sin(2t)}$$

(c) *(1 points)* What is the general solution to the non-homogeneous equation?

$$\boxed{y(t) = C_1 e^{6t} + C_2 e^{-2t} + 2\cos(2t) - 4\sin(2t)}$$

(d) *(5 points)* Find a solution $y(t)$ to the initial value problem

$$y'' - 4y' - 12y = 80 \sin t, \qquad y(0) = 2, \qquad y'(0) = 0.$$

Should be $\sin 2t$.

$y(0) = C_1 + C_2 + 2 = 2 \longrightarrow C_1 + C_2 = 0 \longrightarrow C_1 = -C_2$

$y'(t) = 6C_1 e^{6t} - 2C_2 e^{-2t} - 4\sin(2t) - 8\cos 2t$

$y'(0) = 6C_1 - 2C_2 - 8 = 0 \longrightarrow 3C_1 - C_2 = 4$

$3C_1 + C_1 = 4$

$4C_1 = 4$

$C_1 = 1 \rightarrow C_2 = -1$

$$\boxed{y(t) = e^{6t} - e^{-2t} + 2\cos(2t) - 4\sin(2t)}$$

7. *(2 points each)* Determine if each statement is true or false, and write the word TRUE or FALSE in the space provided. You do not have to justify your answer, and no partial credit will be awarded.

   (a) Let $\vec{v}, \vec{w}, \vec{z}$ be vectors in a real inner product space V. If $\langle \vec{v}, \vec{w} \rangle = 0$ and $\langle \vec{v}, \vec{z} \rangle = 0$, then $\langle \vec{w}, \vec{z} \rangle = 0$.

   FALSE

   (b) If $\langle \vec{v}, \vec{z} \rangle = \langle \vec{w}, \vec{z} \rangle$ for all $\vec{z} \in V$, then $\vec{v} = \vec{w}$.

   TRUE

   (c) The eigenvalues of an orthogonal matrix are all real.

   FALSE

   (d) For any matrix A, the matrix AA^T is diagonalizable.

   TRUE

   (e) The function $y = t\sin(t)$ is a solution to $y^{(4)} + 2y'' + y = 0$.

   TRUE

   (f) Suppose $y = C_1 y_1(t) + C_2 y_2(t)$ is a general solution of a certain second order linear equation $ay'' + by' + cy = 0$. Then the Wronskian $W[y_1, y_2](t) = 0$.

   FALSE

   (g) Suppose $y_1(t)$ is a solution of a certain second order linear equation $ay'' + by' + cy = g(t)$. Then $y_2(t) = Cy_1(t)$, where $C \in \mathbb{R}$ is any constant, is always another solution of the same equation.

   FALSE

8. *(3 points each)* For each part, circle the correct answer. If it is not clear which choice you have circled, no credit will be given. You do not have to justify your answer, and no partial credit will be given.

   (a) Suppose the characteristic equation of a sixth order constant coefficient homogeneous linear equation is

   $$4(r-2)(2+r)^2(1-r)^3 = 0.$$

   Which of the following is *not* a solution of the equation?

   i. $y = 0$

   ii. $y = -t^3e^t$

   iii. $y = 3e^{2t} - 8te^{-2t}$

   iv. $y = 2t^2e^t + 5te^{-2t+4}$

   (b) Which of the following is a solution to the initial value problem

   $$y'' + 2y = e^t, \qquad y(0) = 0, \qquad y'(0) = 0?$$

   i. $y(t) = 0$

   ii. $y(t) = e^t$

   iii. $y(t) = \frac{1}{3}e^t$

   iv. None of the above.

   (c) Which of the following is the correct form of a particular solution to

   $$y^{(4)} - 16y = 2e^{-2t} + 3e^{4t} + \cos(2t) + 5?$$

   i. $A_1e^{-2t} + A_2e^{4t} + A_3\sin(2t) + A_4\cos(2t) + A_5$

   ii. $A_1t^2e^{-2t} + A_2e^{4t} + A_3\sin(2t) + A_4\cos(2t) + A_5$

   iii. $A_1te^{2t} + A_2e^{4t} + A_3t\cos(2t) + A_4t\sin(2t) + A_5$

   iv. None of the above.

Blank page.

Blank page.