Complete all problems. You must show your work or justify your answer for all problems. *Answers without work or justification will receive no credit (even if they are correct).* You may quote theorems and results from class or the homework without justification (name the theorem or state "we proved in class that ..."). Every other fact must be justified. If you need more space, use the blank pages at the back of the exam. If you want me to grade work done any of those pages, clearly indicate this next to the appropriate problem.

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NAME SOLUTIONS

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1. *(2 points each)* Determine if each statement is true or false, and write the word TRUE or FALSE in the space provided. You do not have to justify your answer, and no partial credit will be awarded.

   (a) If A and B are symmetric matrices, then AB must be symmetric as well. (Recall, A is symmetric if $A^T = A$.)

   FALSE

   (b) If an $n \times n$ matrix A does not have n distinct eigenvalues, then A is not diagonalizable.

   FALSE

   (c) If the determinant of a 4×4 matrix A is 4, then $\text{rank}(A) = 4$.

   TRUE

   (d) If A is an $n \times n$ matrix, then the determinant of A is equal to the product of its diagonal entries.

   FALSE

   (e) If A and B are similar matrices, then $\det(A) = \det(B)$.

   TRUE

   (f) If an $n \times n$ matrix A is diagonalizable, then there is a unique diagonal matrix that is similar to A.

   FALSE

   (g) If 3 is an eigenvalue of A, then 9 must be an eigenvalue of A^2.

   TRUE

   (h) The set of all polynomials p of degree at most 5 satisfying $p(0) = p(-2)$ is a vector space.

   TRUE

2. *(4 points)* Let A be a 3×3 matrix with the property that the linear transformation T defined by $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^3$ onto $\mathbb{R}^3$. Explain why the transformation must be injective.

T is surj $\iff$ $\mathrm{Col}(A) = \mathbb{R}^3 \overset{(\mathrm{IMT})}{\iff}$ A is invertible $\underset{(\mathrm{IMT})}{\iff}$ $\mathrm{Nul}(A) = \{\vec{0}\}$ $\iff$ T is injective.

3. *(4 points)* Under what conditions on $a, b, c \in \mathbb{R}$ is the matrix $\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ invertible?

$$\det\begin{pmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = a(2-6) - b(1-9) + c(2-6)$$
$$= -4a + 8b - 4c.$$

Matrix invert. $\iff$ $a - 2b + c \neq 0$.

4. *(4 points)* Give an example of a matrix that is diagonalizable but not invertible.

Many possibilities: $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, for example.

5. *(4 points)* If A is an invertible matrix with eigenvalue $\lambda \neq 0$ and associated eigenvector $\vec{v}$, is $\vec{v}$ an eigenvector of A^{-1}? If so, what is the corresponding eigenvalue? If not, explain why not.

Yes: $A\vec{v} = \lambda\vec{v} \Rightarrow \vec{v} = A^{-1}\lambda\vec{v} \Rightarrow$
$\frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$, so $\vec{v}$ is an e-vect w/ corresp. e-val. $1/\lambda$.

6. Suppose that A is a diagonalizable matrix with characteristic polynomial

$$\chi(\lambda) = \lambda^2(\lambda - 3)(\lambda + 2)^2(\lambda - 4)^3.$$

(a) *(2 points)* Find the size of the matrix A.

A is 8×8, b/c there are 8 e-vals, counted w/ multiplicities.

(b) *(3 points)* Find the dimension of the eigenspace corresponding to $\lambda = 4$.

A is diagonalizable, so geom. mult. is equal to the alg. mult. of $\lambda = 4$, & thus the dimension is 3.

(c) *(3 points)* Find the dimension of Nul(A).

dim Nul (A) is the dimension of the e-space associated to $\lambda = 0$. As in (b), the dimension is 2.

7. *(4 points each)* Find the determinant of each of the following matrices. You must either show your calculations or justify your answer. An answer alone will receive no credit.

(a) $A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$

$$A \sim \begin{pmatrix} 2 & 1 & 1 & 1 \\ 0 & 3/2 & 1/2 & 1/2 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3/2 \end{pmatrix}, \text{ so}$$

$$\det A = (2)(3/2)(2)(3/2) = 9.$$

(b) $B = \begin{pmatrix} 2 & 1 & 3 & -3 \\ 3 & 1 & 4 & 2 \\ 9 & -1 & 8 & 9 \\ 6 & 0 & 6 & 7 \end{pmatrix}$

Column 3 = Column 1 + Column 2, so $\det B = 0$.

8. Let

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix}.$$

(a) *(4 points)* Find all eigenvalues of A.

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & 3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = -\lambda\left[(-1-\lambda)(-1-\lambda) - 1\right]$$

$$= -\lambda\left[\lambda^2 + 2\lambda\right]$$

$$= -\lambda^2(\lambda+2) = 0$$

$$\lambda = 0, \ \lambda = -2$$

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix}.$$

(b) *(5 points)* Find a basis for the eigenspace associated to each eigenvalue.

$\underline{\lambda = 0:}$

$$\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A\vec{x} = \vec{0} \Rightarrow \vec{x} = t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \text{basis: } \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

$\underline{\lambda = -2}:$

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 3 \\ 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A + 2I)\vec{x} = \vec{0} \Rightarrow \vec{x} = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{basis: } \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(c) *(2 points)* Is A diagonalizable? If so, find an invertible matrix P and a diagonal matrix D such that $A = P^{-1}DP$. If not, explain why not.

No, the alg. mult of $\lambda = 0$ is 2, but its geom. mult. is 1, so A is not diagonalizable.

9. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}.$$

(a) *(4 points)* Find a matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^2$.

$T(\vec{x}) = A\vec{x} \Rightarrow A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\5\end{pmatrix}$ & $A\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, so

$A\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix} = \begin{pmatrix}3 & 0\\5 & 1\end{pmatrix}$. $\Rightarrow A = \begin{pmatrix}3 & 0\\5 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\1 & 2\end{pmatrix}^{-1}$

$\Rightarrow A = \frac{1}{3}\begin{pmatrix}3 & 0\\5 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\-1 & 1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}6 & 3\\9 & 6\end{pmatrix} = \boxed{\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}}$

(b) *(5 points)* Given the basis $\mathcal{B} = \left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}\right\}$, find the matrix B such that $T([\vec{x}]_{\mathcal{B}}) = B[\vec{x}]_{\mathcal{B}}$.

$B = P^{-1}AP$, where $P = \begin{pmatrix}1 & 3\\-1 & 3\end{pmatrix}$ & $P^{-1} = \frac{1}{6}\begin{pmatrix}3 & -3\\1 & 1\end{pmatrix}$

$B = \frac{1}{6}\begin{pmatrix}3 & -3\\1 & 1\end{pmatrix}\begin{pmatrix}2 & 1\\3 & 2\end{pmatrix}\begin{pmatrix}1 & 3\\-1 & 3\end{pmatrix}$

$= \frac{1}{6}\begin{pmatrix}3 & -3\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 9\\1 & 15\end{pmatrix}$

$= \frac{1}{6}\begin{pmatrix}0 & -18\\2 & 24\end{pmatrix}$

$= \boxed{\begin{pmatrix}0 & -3\\1/3 & 4\end{pmatrix}}$

(c) *(2 points)* Find the $\mathcal{B}$–coordinates of the vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

$$\left[\begin{pmatrix} 2 \\ 5 \end{pmatrix}\right]_{\mathcal{B}} = P^{-1} \cdot \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -9 \\ 7 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} -2/3 \\ 7/6 \end{pmatrix}}$$

10. *(5 points each)* Let V be the vector space consisting of functions of the form

$$\alpha e^{2x} \cos x + \beta e^{2x} \sin x.$$

Consider the following linear transformation $L\colon V \to V$ defined by

$$L(f) = f' + f,$$

where f' is the derivative of a vector $f \in V$.

(a) Find the matrix of L with respect to the basis $\{e^{2x} \cos x, e^{2x} \sin x\}$ of V.

$$L(e^{2x}\cos x) = 2e^{2x}\cos x - e^{2x}\sin x + e^{2x}\cos x$$
$$= 3e^{2x}\cos x - e^{2x}\sin x$$

$$L(e^{2x}\sin x) = 2e^{2x}\sin x + e^{2x}\cos x + e^{2x}\sin x$$
$$= 3e^{2x}\sin x + e^{2x}\cos x$$

$$\boxed{\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}}$$

(b) Use your answer from (a) to find a function f such that $f' + f = e^{2x} \cos x$.

$$\begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix} \vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{1}{10} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$= \frac{1}{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3/10 \\ 1/10 \end{pmatrix}$$

$$\Rightarrow \boxed{f(x) = \frac{3}{10} e^{2x} \cos x + \frac{1}{10} e^{2x} \sin x}$$

11. *(10 points each)* Choose TWO of the following problems. Circle the problems you would like me to grade.

$(a) \qquad (b) \qquad (c)$

I will not look at the other problem. If you leave this blank, or it is not clear, I will grade the first two problems. But sure to *completely* justify your solution.

(a) Let $A \in M_{m,n}(\mathbb{R})$ and $B \in M_{n,p}(\mathbb{R})$. Show that

$$\operatorname{rank}(AB) \le \operatorname{rank}(A) \operatorname{rank}(B).$$

$\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$, b/c columns of AB are a lin comb. of the columns of A.

$\Rightarrow \dim(\mathrm{Col}(AB)) \le \dim(\mathrm{Col}(A))$

$\Rightarrow \operatorname{rank}(AB) \le \operatorname{rank}(A)$

- if $\operatorname{rank}(B) \geq 1$, then

$\operatorname{rank}(AB) \le \operatorname{rank}(A) \le \operatorname{rank}(A)\operatorname{rank}(B)$.

- if $\operatorname{rank}(B) = 0$, then

$\dim(\mathrm{Col}(B)) = 0 \Rightarrow \forall \vec{x} \in \mathbb{R}^p,\ B\vec{x} = \vec{0} \Rightarrow$

$AB\vec{x} = A(B\vec{x}) = A\cdot\vec{0} = \vec{0}$

so $\operatorname{rank}(AB) = 0$ &

$0 = \operatorname{rank}(AB) \le \operatorname{rank}(A)\operatorname{rank}(B) = \operatorname{rank}(A)\cdot 0 = 0.$

(b) Let A, B be $n \times n$ matrices such that $AB = A^T$. Prove that the row space of A is equal to its column space.

$\mathrm{Row}(A) = \mathrm{Col}(A^T) = \mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$.

But $\dim(\mathrm{Row}\,A) = \dim(\mathrm{Col}(A))$ by Rank-Nullity. If a subspace of a v.s. has the same dim. as the v.s., then they are equal, so $\mathrm{Row}(A) = \mathrm{Col}(A)$.

(c) Let A be an invertible $n \times n$ matrix and let B be an $n \times m$ matrix. Show that $\dim(\text{Nul}(B)) = \dim(\text{Nul}(AB))$ and $\dim(\text{Col}(B)) = \dim(\text{Col}(AB))$.

$\vec{x} \in \text{Nul}(AB) \Rightarrow AB\vec{x} = \vec{0} \Rightarrow B\vec{x} = A^{-1}\vec{0} = \vec{0}$

$\Rightarrow \vec{x} \in \text{Nul}(B)$.

$\vec{x} \in \text{Nul}(B) \Rightarrow B\vec{x} = \vec{0} \Rightarrow AB\vec{x} = A\vec{0} = \vec{0}$

$\Rightarrow x \in \text{Nul}(AB)$.

Therefore, $\text{Nul}(B) = \text{Nul}(AB)$, so their dimensions are equal.

By rank-nullity,

$\dim(\text{Nul}(B)) + \dim(\text{Col}(B)) = n$ &

$\dim(\text{Nul}(AB)) + \dim(\text{Col}(AB)) = n$.

Thus $n - \dim(\text{Col}(B)) = n - \dim(\text{Col}(AB))$,

& so $\dim(\text{Col}(B)) = \dim(\text{Col}(AB))$.

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