MATH 53 SECOND MIDTERM EXAM, PROF. SRIVASTAVA
APRIL 12, 2018, 5:10PM–6:30PM, 155 DWINELLE HALL.

Name: Nikhil Srivastava

SID: ______

GSI: ______

NAME OF THE STUDENT TO YOUR LEFT: ______
NAME OF THE STUDENT TO YOUR RIGHT: ______

INSTRUCTIONS: Write all answers clearly in the provided space. This exam includes some space for scratch work at the bottom of pages 2 and 4 which will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. **Show your work** — numerical answers without justification will be considered suspicious and will not be given full credit. Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.
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Sign here: ______

Question	Points
1	14
2	14
3	12
4	8
5	14
6	12
7	12
8	14
Total:	100

Do not turn over this page until your instructor tells you to do so.

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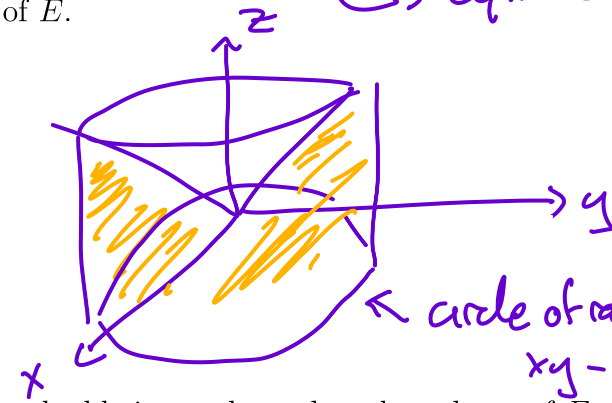
1. Let E be the solid region bounded from above by the surface $z^2 = x^2 + y^2$, from below by the plane $z = 0$, and from the sides by the surface $x^2 + y^2 = 1$.

cone

cylinder of radius 1

(a) (4 points) Draw a rough sketch of E.



(b) (10 points) Set up and evaluate a double integral equal to the volume of E.

Vol(E) = volume under the graph of $z = \sqrt{x^2+y^2}$

above the domain $D = \{(x,y) : x^2+y^2 \leq 1\}$.

$= \iint_D \sqrt{x^2+y^2}\, dA$. Since D and the function are radially symmetric, polar coordinates should help.

$D = \{(r,\theta) : 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1\}$

$\sqrt{x^2+y^2} \longrightarrow \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

$dA \longrightarrow r\,dr\,d\theta$

So $\text{Vol} = \int_0^{2\pi}\int_0^1 r \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{r^3}{3}\Big|_0^1 d\theta = \frac{1}{3} \cdot 2\pi = 2\pi/3$

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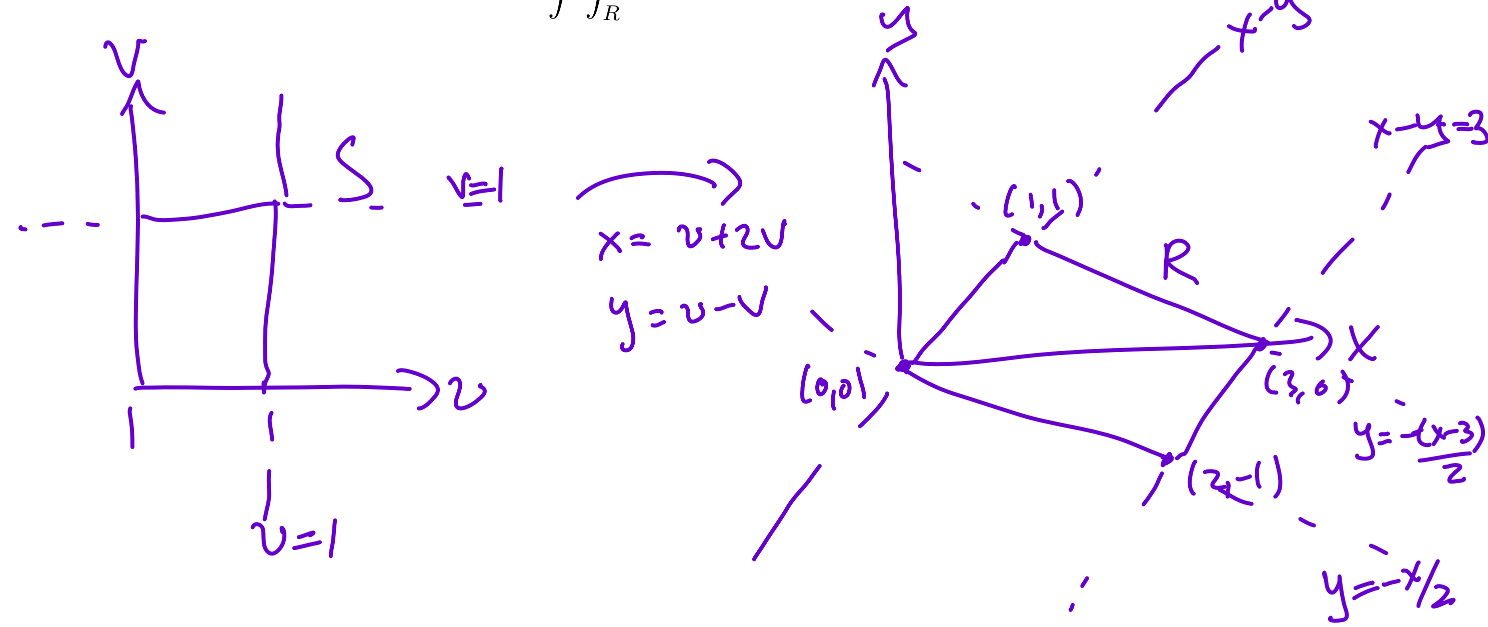
2. (14 points) Let R be the parallelogram with vertices $(0,0), (1,1), (2,-1)$, and $(3,0)$. Use the change of variables

$$x = u + 2v, \quad y = u - v$$

to evaluate the integral

$$\int\int_R (x+2y)^2 e^{x-y} dA.$$



the 4 lines defining the boundary become:

$x-y=0 \longrightarrow u+2v-(u-v)=3v=0 \Longrightarrow v=0$

$x-y=1 \longrightarrow u+2v-(u-v)=3 \Longrightarrow v=1$

$y+\frac{x}{2}=0 \Longrightarrow u-v+\frac{u+2v}{2}=0 \Longrightarrow u=0$

$y+\frac{x-3}{2}=0 \Longrightarrow u-v+\frac{u+2v-3}{2}=0 \Longrightarrow u=1$

So the parallelogram R corresponds to the unit square. The jacobian is

$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = -1-2$, so we have

(contd on Scratch Paper 1)

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3. (12 points) Find the average distance from a point in a ball of radius a (i.e., a solid sphere in $\mathbb{R}^3$ centered at the origin) to its center. Let E = sphere of radius a at origin.

The required average is:

$$\frac{\iiint_E \text{dist}(x,y,z)\, dV}{\iiint_E 1\, dV}$$ where $\text{dist}(x,y,z) = \sqrt{x^2+y^2+z^2}$.

Since dist and E are spherically symmetric, we switch to spherical coords:

$$\text{dist}(x,y,z) \longrightarrow \rho$$

$$E \longrightarrow \{0 \le \rho \le a, 0 \le \varphi \le \pi, 0 \le \theta \le 2\pi\}.$$

We now have:

$$\iiint_E 1\, dV = \frac{4}{3}\pi a^3 \quad \text{(done in class)}$$

$$\iiint_E \rho\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\varphi \left.\frac{\rho^4}{4}\right|_0^a d\varphi\, d\theta = \frac{a^4}{4}\int_0^{2\pi} \left(-\cos\varphi\right)\Big|_0^{\pi} d\theta$$

$$= \frac{2a^4}{4}\int_0^{2\pi} d\theta = \pi a^4.$$ So the average dist is

$$\frac{\pi a^4}{\frac{4}{3}\pi a^3} = \frac{3}{4}a.$$

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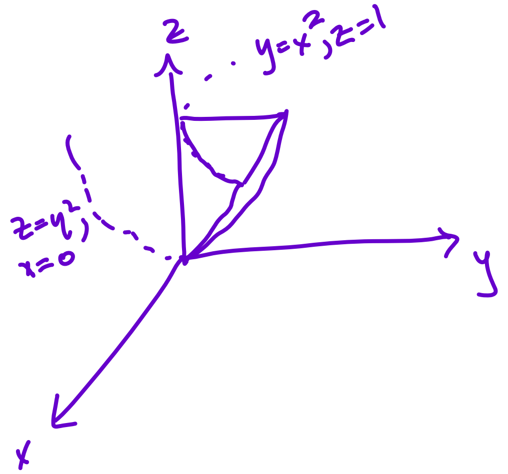
4. (8 points) Change the order of integration:

$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz = \int_?^? \int_?^? \int_?^? f(x,y,z)\,dy\,dz\,dx.$$

The limits imply: $0 \le z \le 1$, $\underbrace{0 \le y \le \sqrt{z}}_{z \ge y^2}$, $\underbrace{0 \le x \le \sqrt{y}}_{y \ge x^2}$



range of x: 0 to 1 (by setting y, z = 1)

range of z given x: $1 \ge z \ge y^2 \ge x^4$

range of y given z and x: $x^2 \le y \le \sqrt{z}$

So limits: $\displaystyle\int_0^1 \int_{x^4}^1 \int_{x^2}^{\sqrt{z}} f(x,y,z)\,dy\,dz\,dx$

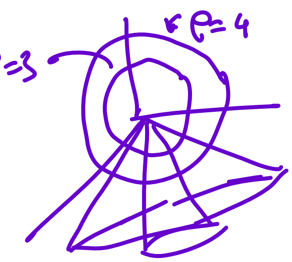
5. (a) (6 points) Let E be the region in $\mathbb{R}^3$ bounded by the sphere of radius 3 at the origin, the sphere of radius 4 at the origin, the cone $z = -\sqrt{x^2+y^2}$, and the cone $z = -2\sqrt{x^2+y^2}$. Which of the following integrals represents the volume of E? (circle exactly one)

~~1.~~ $\int_3^4 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{x^2+y^2}}^{-2\sqrt{x^2+y^2}} dz\,dy\,dx$ x-projection is not [3,4]

Not a volume ~~2.~~ $\int_3^4 \int_0^{2\pi} \int_{3\pi/4}^{5\pi/6} d\phi\,d\theta\,d\rho$

~~3.~~ $\int_3^4 \int_0^{2\pi} \int_{-2r^2}^{-r^2} r\,dz\,dr\,d\theta$ Projection on xy-plane is not annulus (3,4)

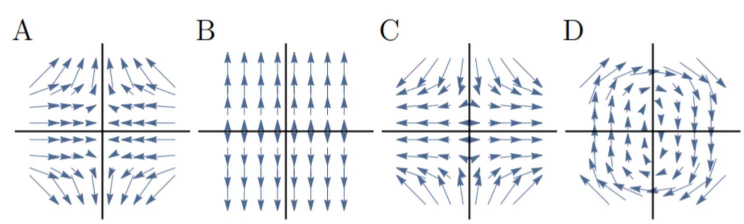
4. None of the above.



(b) (8 points) Match the vector field to the picture. There is an exact match.

Field	A-D
$\vec{F}(x,y) = \langle 0, y \rangle$	B
$\vec{F}(x,y) = \langle -x, y^3 \rangle$	A
$\vec{F}(x,y) = \langle y^3, -x \rangle$	D
$\vec{F}(x,y) = \langle x, -y^3 \rangle$	C

opposites of each other. distinguish by looking at x-coordinate.



[Scratch Paper 1]

$$\iint_R (x+2y)^2 e^{x-y}\,dx\,dy \qquad \left(\begin{array}{l} x = u+2v \\ y = u-v \\ \text{so } x+2y = 3u \\ \quad x-y = 3v \end{array} \right.$$

$$= \iint_S (3u)^2 e^{3v} \, |-3| \, du\,dv$$

$$= 3\int_0^1\int_0^1 9u^2 e^{3v}\,du\,dv = 27\int_0^1 \left.\frac{u^3}{3}\right|_0^1 e^{3v}\,dv$$

$$= 9 \cdot \left.\frac{e^{3v}}{3}\right|_0^1 = \underline{\underline{3(e^3-1)}}$$

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6. Consider the vector field $\mathbf{F} = \langle 4x\ln(y), \frac{2x^2-1}{y}\rangle$ defined on $D = \{(x,y) : y > 0\}$.

(a) (4 points) Explain why $\mathbf{F}$ is conservative.

$$\text{Curl}(F) = \frac{\partial}{\partial x}\left(\frac{2x^2-1}{y}\right) - \frac{\partial}{\partial y}\left(4x\ln(y)\right)$$

$$= \frac{4x}{y} - \frac{4x}{y} = 0.$$ Since D is simply connected, F is conservative.

(b) (8 points) Using a systematic method, find a potential $f(x,y)$ defined on D such that $\mathbf{F} = \nabla f$.

We want solve: $f_x = 4x\ln(y)$ ①, $f_y = \frac{2x^2-1}{y}$ ②

Integrating ①:

$$f(x,y) = \int 4x\ln(y)\,dx = 2x^2\ln(y) + g(y)$$

for some unknown $g(y)$.

To find g, we plug this into ②:

$$\frac{\partial}{\partial y}\left(2x^2\ln(y) + g(y)\right) = \frac{2x^2}{y} + g'(y) = \frac{2x^2-1}{y}.$$

Thus $g'(y) = \frac{-1}{y} \Rightarrow g(y) = -\ln(y) + C$

So $f(x,y) = \underline{2x^2\ln(y)} - \ln(y) + C$, for C constant.

7. (12 points) Calculate the work done by the force field $\mathbf{F} = \langle y^2+1, 2xy+\sin^6(y)\rangle$ on a particle moving in the plane with trajectory $\mathbf{r}(t) = \langle e^t, (t^6-1)\sin(t)\rangle$, $t \in [0,1]$.

The path given looks complicated, so it would be nice if the integral was independent of path (in which case we could replace by a simpler path).

Let's check this: $\text{curl}(F) = 2y - 2y = 0$, and F is defined on $\mathbb{R}^2$ which is simply connected, so indeed the integral is independent of path!

The endpoints of the given path are

$$\vec{r}(0) = \langle 1, 0\rangle, \quad \vec{r}(1) = \langle e, 0\rangle.$$

So instead, let's integrate along C: $\vec{r}(t) = \langle t, 0\rangle$, $t \in [1, e]$.

$\vec{r}\,'(t) = \langle 1, 0\rangle$

which is

$$\int_1^e F(t,0)\cdot\langle 1,0\rangle\,dt$$

$$= \int_1^e \langle 1,0\rangle\cdot\langle 1,0\rangle\,dt = \underline{\underline{e-1}}.$$

8. Let $\mathbf{F} = \langle 2xy, 3xy \rangle$, and let C be a positively oriented unit circle centered at the origin. Use Green's theorem to evaluate:

   (a) (7 points) The work $\int_C \mathbf{F} \cdot d\mathbf{r}$.

$$\text{Curl}(F) = \frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(2xy) = 3y - 2x.$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (3y - 2x)\, dA \quad \text{where } D \text{ is the unit circle}$$

$$= 3\iint_D y\,dA - 2\iint_D x\,dA = 0 - 0 = 0$$

Since the center of mass of a circle with unit density is its center, which is (0,0).

   (b) (7 points) The flux $\int_C \mathbf{F} \cdot \mathbf{n} ds$.

$$\text{div}(F) = 2y + 3x.$$

$$\int_C \vec{F} \cdot \vec{n}\, ds = \iint_D 2y + 3x \; dA$$

Could also use symmetry about x-axis and y-axis.

$$= 2\iint_D y\,dA + 3\iint_D x\,dA = 0 + 0 = \underline{0}$$

by the same reasoning as above.

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[Scratch Paper 2]