Midterm Solutions

March 14, 2018

The duration is 80 minutes. Each problem is worth 25 points. Closed book/notes; one formula sheet allowed. Answers without justification do not receive full credit.

1. Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + \frac{\mu}{1+x_2}\\ \dot{x}_2 &= -x_2 + \frac{\mu}{1+x_1},\end{aligned}$$

where $\mu > 0$ is a positive parameter.

a) Show that the nonnegative quadrant $\mathbb{R}^2_{\geq 0}$ is positively invariant.

Solution: We can show that the nonnegative quadrant is positively invariant by individually proving that the half spaces $x_1 \geq 0, x_2 \geq 0$ are both positively invariant. This can be done by proving a) that $\dot{x}_1 \geq 0$ whenever $x_1 = 0, x_2 \geq 0$ and b) $\dot{x}_2 \geq 0$ whenever $x_2 = 0, x_1 \geq 0$. When $x_1 = 0$ and $x_2 \geq 0$,

$$\dot{x}_1 = \frac{\mu}{1+x_2} > 0$$

which proves invariance of the $x_1 \geq 0$ half space. A symmetric argument can be made for invariance of the $x_2 \geq 0$ half space.

b) Show that a single equilibrium exists in the nonnegative quadrant.

Solution:

$$\begin{aligned}0 = -x_1 + \frac{\mu}{1+x_2} &\Rightarrow x_1 = \frac{\mu}{1+x_2}\\ 0 = -x_2 + \frac{\mu}{1+x_1} &\Rightarrow x_2 = \frac{\mu}{1+x_1}\end{aligned}$$

Substituting the expression for x_1 into the x_2 equation yields:

$$x_2 = \frac{\mu}{1+\frac{\mu}{1+x_2}} = \frac{\mu(1+x_2)}{1+x_2+\mu}$$

Multplying both sides by $1 + x_2 + \mu$ yields

$$\begin{aligned}x_2 + x_2^2 + \mu x_2 &= \mu + \mu x_2\\ &\Rightarrow x_2^2 + x_2 - \mu = 0\end{aligned}$$

The quadratic equation gives us two possible solutions

$$x_2 = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\mu}$$

When $\mu > 0$, both solutions are real and only one solution exists in the nonnegative orthant. Noting that the system dynamics are symmetric with respect to swapping x_1 and x_2, we derive the same equilibrium for x_1.

$$(x_1, x_2) = (-\frac{1}{2} + \frac{1}{2}\sqrt{1+4\mu}, -\frac{1}{2} + \frac{1}{2}\sqrt{1+4\mu})$$

c) Determine whether this equilibrium is stable or not using the linearization method. Does your answer depend on the value of μ?

The Jacobian of our system is

$$J(x) = \frac{\partial f}{\partial x}(x) = \begin{bmatrix} -1 & \frac{-\mu}{(1+x_2)^2} \\ \frac{-\mu}{(1+x_1)^2} & -1 \end{bmatrix}$$

The trace of the Jacobian is -2 so there must exist at least one negative eigenvalue. We use the determinant to distinguish between a stable and a saddle point.

$$\det(J(x)) = 1 - \frac{\mu^2}{(1+x_1)^2(1+x_2)^2}$$

Substituting in the equilibrium point from part 1b yields:

$$1 - \frac{\mu^2}{(\frac{1}{2} + \frac{1}{2}\sqrt{1+4\mu})^4} = 1 - \frac{\mu^2}{(\frac{1}{2} + \sqrt{\frac{1}{4}+\mu})^4} > 1 - \frac{\mu^2}{\mu^2} = 0$$

The strict inequality holds because shrinking the denominator of the subtracted term causes the term to grow and $\sqrt{\mu}^4 = \mu^2$. As long as $\mu > 0$ the determinant is strictly positive so both eigenvalues must be the same sign. *The equilibrium point is stable and its stability characteristics do not depend on the value of μ.*

d) Determine whether any periodic orbits exist in the nonnegative quadrant. Explain your reasoning.

Solution: The system is time invariant and planar. Moreover, the divergence is not identically zero and does not change sign in the nonnegative quadrant.

$$\nabla \cdot f(x) = -1 - 1 = -2$$

Invoking Bendixson's theorem implies that there are *no periodic orbits* that exist in the nonnegative quadrant.

2. a) For the matrix A below find $P = P^T > 0$ such $A^T P + PA$ is negative semidefinite:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}.$$

Solution: Consider the P matrix

$$P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Substituting in to $A^T P + PA$ yields:

$$A^T P + PA = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The zero matrix is negative semidefinite. One way to notice that this particular P satisfies the inequality is to plot the phase portrait for the system and see that there is an energy function that is conserved.

b) Does there exist $P = P^T > 0$ such $A^T P + PA$ is negative definite? If your answer is yes, produce such a P; otherwise explain why none exists.

Solution: No, there does not exist a matrix $P = P^T > 0$ such that $A^T P + PA$ is negative definite. Existence of such a P would imply that the system $\dot{x} = Ax$ is asymptotically stable. The eigenvalues of A are on the imaginary axis so A is stable but not asymptotically stable.

$$0 = \det(sI - A) = \det\left(\begin{bmatrix} s & -1 \\ 2 & s \end{bmatrix}\right) = s^2 + 2 \Rightarrow s = \pm\sqrt{2}i$$

3. a) Show that the origin is globally asymptotically stable for the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - x_1^3 - x_1^5. \end{aligned}$$

Solution:

Consider the following candidate Lyapunov function

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{6}x_1^6 + \frac{1}{2}x_2^2.$$

This function is positive definite because if either x_1 or x_2 are non-zero then $V(x) > 0$. Moreover, it is radially unbounded because $||x||_2 \to \infty$ implies that either $|x_1| \to \infty$ or $|x_2| \to \infty$. Both would imply $V(x) \to \infty$.

$$\begin{aligned} \dot{V} &= (x_1^3 + x_1^5)(x_2) + (x_2)(-x_2 - x_1^3 - x_1^5) \\ &= x_1^3 x_2 + x_1^5 x_2 - x_2^2 - x_1^3 x_2 - x_1^5 x_2 \\ &= -x_2^2 \\ &\leq 0 \end{aligned}$$

While $\dot{V}$ is only negative semidefinite, we can invoke the Lasalle-Krasovskii principle to prove global asymptotic stability. Let $S = \{x : \dot{V} = 0\} = \{x : x_2 = 0\}$. Within this region, the system dynamics are

$$\begin{aligned}\dot{x}_1 &= 0 \\ \dot{x}_2 &= -x_1^3 - x_1^5\end{aligned}$$

The biggest invariant set in S is the origin. Any other point such that $x_1 \neq 0$ will cause the system to exit S. The system is globally asymptotically stable.

b) Is it also exponentially stable? Explain your reasoning.

Solution: The proposition in Lecture 11 Page 1 says that the origin is exponentially stable for $\dot{x} = f(x), f(0) = 0$ if and only if $A = \frac{\partial f}{\partial x}\big|_{x=0}$ is Hurwitz and all of A's eigenvalues have strictly negative real components.

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ -3x_1^2 - 5x_1^4 & -1 \end{bmatrix}\bigg|_{x=0} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Because A is not full rank, it must contain a zero eigenvalue. A is therefore not Hurwitz, and the system is not exponentially stable.

4. Determine whether each system below is input-to-state stable with respect to u. Justify your answer in each case.

a) $\dot{x} = -x - xu^2$

Solution: The system is ISS and satisfies the following ISS inequality with $\gamma = 0$.

$$|x(t)| \leq |x(0)|e^{-t}$$

More informally, any nonzero u will only cause x to approach the origin at a *faster* rate than the system $\dot{x} = -x$.

b) $\dot{x}_1 = -x_1 + x_1x_2, \quad \dot{x}_2 = -x_2 + u$

Solution: No, this system is not ISS. Consider the input where $u(t) = x_2(0)$ for all t. This input u is bounded and $\dot{x}_2 = 0$ for all t. If $x_2(0) > 1$ then $|x_1(t)|$ is lower bounded by a term $e^{(x_2(0)-1)t}|x_1(0)|$ that grows to infinity.

c) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + u.$

Solution: No, this system is not ISS. This system exhibits a circular orbit around the origin. Even with $u = 0$, no class-$\mathcal{KL}$ function β exists such that $|x(t)| \leq \beta(x(0), t)$ because the function cannot decay to zero as $t \to \infty$. Alternatively, one could also drive the system at its resonant frequency to create an unstable trajectory with a bounded u.