Second Midterm Examination
Monday April 9, 2018
Closed Books and Closed Notes

Question 1 *Dynamics of a Dumbbell* *(25 Points)*

As shown in Figure 1, a particle of mass m_1 and a second particle of mass m_2 are connected together by a massless inextensible rod of length ℓ. In addition, a pair of constant forces $P_1\mathbf{E}_y$ and $-P_2\mathbf{E}_y$ act on the particles as shown in the figure. To describe the kinematics of this system, the position vector of the center of mass C is described using a set of Cartesian coordinates and the position vectors of m_1 and m_2 relative to C are described using a set of cylindrical polar coordinates:

$$\mathbf{r} = x\mathbf{E}_x + y\mathbf{E}_y, \qquad \mathbf{r}_1 - \mathbf{r} = \frac{m_2\ell}{m_1+m_2}\mathbf{e}_r, \qquad \mathbf{r}_2 - \mathbf{r} = -\frac{m_1\ell}{m_1+m_2}\mathbf{e}_r, \tag{1}$$

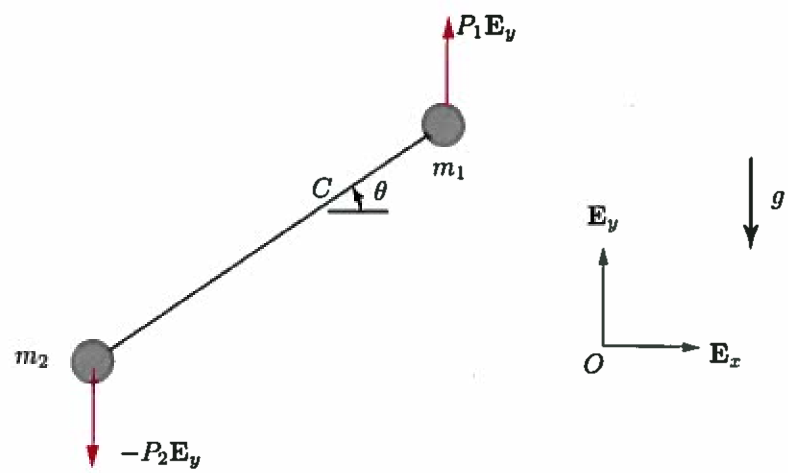


Figure 1: A system of two particles connected by a massless rod of length ℓ. Both particles are free to move on a smooth vertical plane.

(a) *(3+5+2+5 Points)* Starting from the representations (1) and using the definitions of the linear momentum $\mathbf{G}$, angular momentum $\mathbf{H}_C$ relative to the center of mass, angular momentum $\mathbf{H}_O$ relative to the fixed point O, and kinetic energy T, show that

$$\begin{aligned} \mathbf{G} &= m\left(\dot{x}\mathbf{E}_x + \dot{y}\mathbf{E}_y\right), \\ \mathbf{H}_C &= \frac{m_1 m_2}{m}??\mathbf{E}_z, \qquad \mathbf{H}_O = ???\mathbf{E}_z, \\ T &= \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2}\frac{m_1 m_2}{m}????, \end{aligned} \tag{2}$$

where the mass of the system $m = m_1 + m_2$. For full credit, supply the missing terms.

(b) *(5 Points)* Using the work-energy theorem $\dot{E} = \mathbf{F}_{\mathrm{nc}_1}\cdot\mathbf{v}_1 + \mathbf{F}_{\mathrm{nc}_2}\cdot\mathbf{v}_2$, prove that the total energy of the system is conserved. For full credit, supply an expression for E.

(c) *(5 Points)* Show that the motion of the system is governed by the differential equations

$$m\ddot{x} = 0, \qquad m\ddot{y} = P_1 - P_2 - mg, \qquad \frac{m_1 m_2}{m}\ell^2\ddot{\theta} = \frac{\left(P_1 m_2 + P_2 m_1\right)\ell}{m}\cos(\theta). \tag{3}$$

Question 2 *Planar Motion of a System of Three Particles* *(25 Points)*

Referring to Figure 2, a system of three particles is free to move on a vertical plane. The particle of mass m_2 is attached by a linear spring of stiffness K and unstretched length ℓ_0 to a particle of mass m_1. Both m_1 and m_2 are free to move in the y-direction in a frictionless guide. A particle of mass m_3 is attached by a rigid massless rod of length ℓ to m_2 using a pin joint.

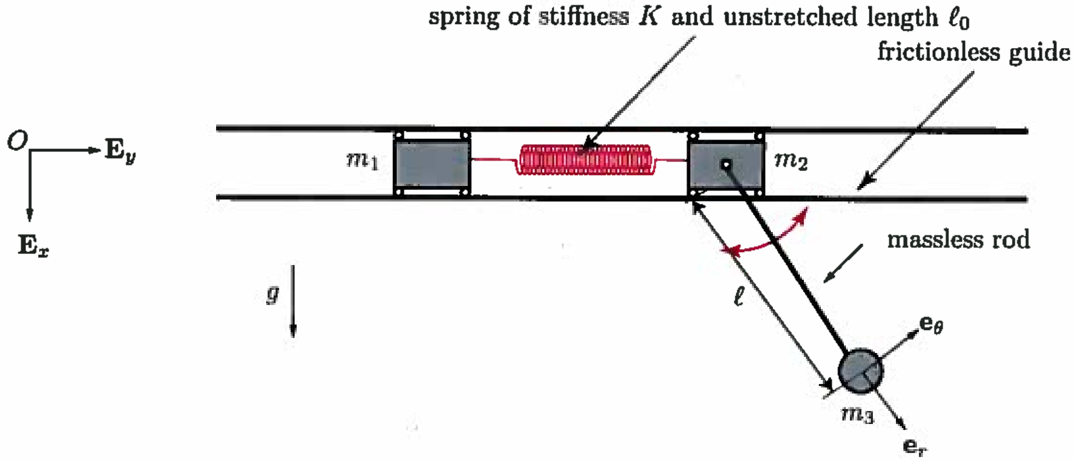


Figure 2: A system of three particles that is in motion on a vertical plane.

(a) *(2+5 Points)* Starting from the representations

$$\mathbf{r}_1 = y_1\mathbf{E}_y, \qquad \mathbf{r}_2 = (y_1 + y_2 + \ell_0)\,\mathbf{E}_y, \qquad \mathbf{r}_3 = (y_1 + y_2 + \ell_0)\,\mathbf{E}_y + \ell\mathbf{e}_r, \tag{4}$$

where $\mathbf{e}_r$ is a unit vector pointing from m_2 to m_3, establish representations for the linear momentum $\mathbf{G}$ and kinetic energy T of the system of particles.

(b) *(9 Points)* Draw freebody diagrams of each of the three particles. For full credit, provide a clear expression for the spring force. You should assume that $y_2 + \ell_0 > 0$.

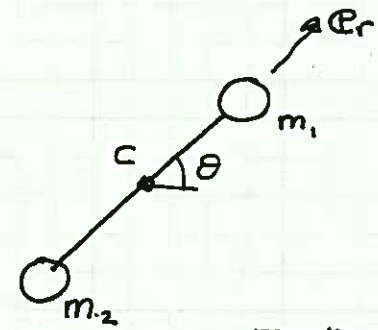
(c) *(3 Points)* Show that the following momentum is conserved during a motion of the system:

$$(m_1 + m_2 + m_3)\,\dot{y}_1 + (m_2 + m_3)\,\dot{y}_2 + m_3\ell\dot{\theta}\cos(\theta)\,. \tag{5}$$

(d) *(6 Points)* Using the work-energy theorem $\dot{T} = \mathbf{F}_1 \cdot \mathbf{v}_1 + \mathbf{F}_2 \cdot \mathbf{v}_2 + \mathbf{F}_3 \cdot \mathbf{v}_3$, show that the total energy E is conserved, where the potential energy of the system is

$$U = \frac{K}{2}y_2^2 - m_3 g\ell\cos(\theta)\,. \tag{6}$$

QUESTION 1



(a) $\underset{\sim}{r} = x\underline{E}_x + y\underline{E}_y$

$\underline{G} = m\dot{\underline{r}} = m\dot{x}\underline{E}_x + m\dot{y}\underline{E}_y$

$\underline{r}_1 = \underline{r} + \frac{m_2 \ell}{m}\mathbf{e}_r, \quad \underline{r}_2 = \underline{r} - \frac{m_1 \ell}{m}\mathbf{e}_r$ $\qquad m = m_1 + m_2$

$$\underline{H}_c = (\underline{r}_1 - \underline{r}) \times m_1(\underline{v}_1 - \underline{v}) + (\underline{r}_2 - \underline{r}) \times m_2(\underline{v}_2 - \underline{v})$$

$$= \frac{+m_2 \ell}{(m_1+m_2)}\mathbf{e}_r \times m_1\left(\frac{m_2 \ell}{m_1+m_2}\dot{\theta}\underline{\mathbf{e}}_\theta\right)$$

$$+ \frac{-m_1 \ell}{m_1+m_2}\mathbf{e}_r \times m_2\left(\frac{-m_1 \ell}{m_1+m_2}\dot{\theta}\underline{\mathbf{e}}_\theta\right)$$

$$= \left(\frac{m_2^2 \ell^2 m_1}{(m_1+m_2)^2} + \frac{m_1^2 m_2 \ell^2}{(m_1+m_2)^2}\right)\dot{\theta}\underline{E}_z$$

$$= \frac{m_1 m_2}{m_1+m_2}\ell^2\dot{\theta}\underline{E}_z$$

$$\underline{H}_o = \underline{r} \times m\underline{v} + \underline{H}_c$$

$$= (x\underline{E}_x + y\underline{E}_y) \times m(\dot{x}\underline{E}_x + \dot{y}\underline{E}_y) + \underline{H}_c$$

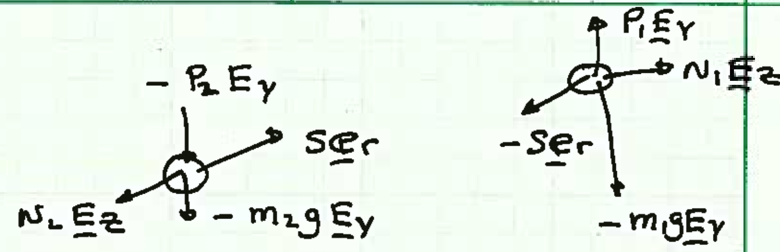
$$= m(x\dot{y} - y\dot{x})\underline{E}_z + \frac{m_1 m_2}{m}\ell\dot{\theta}\underline{E}_z$$

$$T = \frac{1}{2}m\underline{v}\cdot\underline{v} + \frac{1}{2}m_1(\underline{v}_1 - \underline{v})\cdot(\underline{v}_1 - \underline{v}) + \frac{1}{2}m_2(\underline{v}_2 - \underline{v})\cdot(\underline{v}_2 - \underline{v})$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}m_1\ell^2\left(\frac{m_2}{m}\right)^2\dot{\theta}^2 + \frac{1}{2}m_2\ell^2\left(\frac{m_1}{m}\right)^2\dot{\theta}^2$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{m_1 m_2}{m}\ell^2\dot{\theta}^2$$

(b)



$$E = T + (m_2 g + P_2)\mathbf{E}_y \cdot \mathbf{r}_2 + (m_1 g - P_1)\mathbf{E}_y \cdot \mathbf{r}_1$$

$$\begin{aligned}\dot{E} &= N_2 \mathbf{E}_z \cdot \mathbf{v}_2 + N_1 \mathbf{E}_z \cdot \mathbf{v}_1 + S\mathbf{e}_r \cdot (\mathbf{v}_2 - \mathbf{v}_1) \\ &= 0 + 0 + S\mathbf{e}_r \cdot (-\ell\dot{\theta}\mathbf{e}_\theta) \\ &= 0\end{aligned}$$

Hence E is conserved

(c)

From $(\mathbf{F} = \dot{\mathbf{G}}) \cdot \mathbf{E}_x$ $\quad m\ddot{x} = 0$

$(\mathbf{F} = \dot{\mathbf{G}}) \cdot \mathbf{E}_Y$ $\quad m\ddot{y} = P_1 - P_2 - mg$

From $(\mathbf{M}_c = \dot{\mathbf{H}}_c) \cdot \mathbf{E}_z$

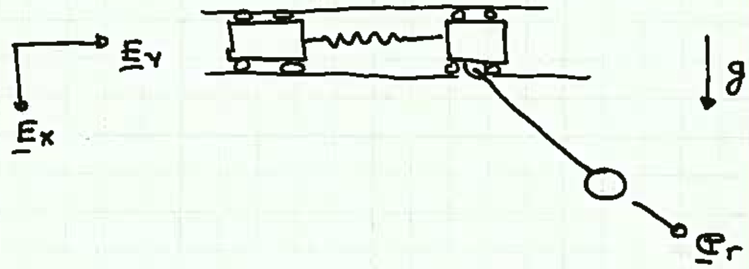
Now $$\mathbf{M}_c \cdot \mathbf{E}_z = \mathbf{E}_z \cdot \left(-\frac{m_1 \ell}{m}\mathbf{e}_r \times -P_2\mathbf{E}_Y + \frac{m_2 \ell}{m}\mathbf{e}_r \times P_1 \mathbf{E}_Y\right)$$

$$= \frac{m_1 \ell}{m} P_2 \cos\theta + \frac{m_2 \ell}{m} P_1 \cos\theta$$

Hence

$$\frac{m_1 m_2}{m}\ell^2\ddot{\theta} = \frac{\ell}{m}\left(P_2 m_1 + P_1 m_2\right)\cos\theta$$

QUESTION 2



$$\underline{r}_1 = y_1 \underline{E}_Y$$

$$\underline{r}_2 = (y_1 + y_2 + \ell_0)\underline{E}_Y$$

$$\underline{r}_3 = \underline{r}_2 + \ell \underline{e}_r$$

(a) $$\underline{G} = m_1 \dot{\underline{r}}_1 + m_2 \dot{\underline{r}}_2 + m_3 \dot{\underline{r}}_3$$

$$= (m_1 + m_2 + m_3)\dot{y}_1 \underline{E}_Y + (m_2 + m_3)\dot{y}_2 \underline{E}_Y + m_3 \ell \dot{\theta} \underline{e}_\theta$$

$$T = \tfrac{1}{2} m_1 \underline{v}_1 \cdot \underline{v}_1 + \tfrac{1}{2} m_2 \underline{v}_2 \cdot \underline{v}_2 + \tfrac{1}{2} m_3 \underline{v}_3 \cdot \underline{v}_3$$

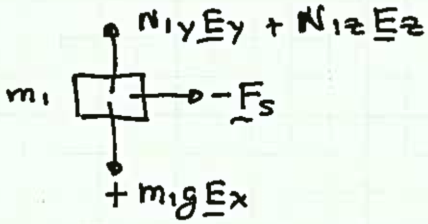
$$= \tfrac{1}{2} m_1 \dot{y}_1^2 + \tfrac{1}{2} m_2 (\dot{y}_1 + \dot{y}_2)^2 + \tfrac{1}{2} m_3 \left(\underline{E}(\dot{y}_1 + \dot{y}_2) + \ell\dot{\theta}\underline{e}_\theta\right) \cdot \left((\dot{y}_1 + \dot{y}_2)\underline{E}_Y + \ell\dot{\theta}\underline{e}_\theta\right)$$

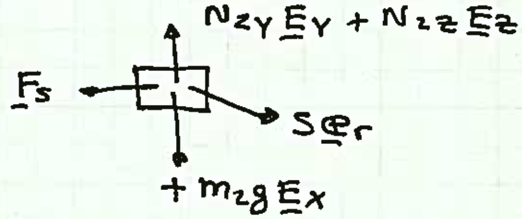
$$= \tfrac{1}{2}(m_1 + m_2 + m_3)\dot{y}_1^2 + \tfrac{1}{2}(m_2 + m_3)\dot{y}_2^2$$

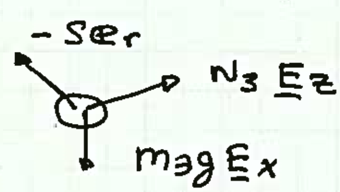
$$+ (m_2 + m_3)\dot{y}_1 \dot{y}_2 + \tfrac{1}{2} m_3 \ell^2 \dot{\theta}^2$$

$$+ m_3 \ell \dot{\theta} (\dot{y}_1 + \dot{y}_2) \cos\theta$$

(b)







Spring force on $m_2 = \underline{F}_s$:

$$\underline{F}_s = -K y_2 \underline{E}_Y$$

$$= -K \left(\|\underline{r}_2 - \underline{r}_1\| - \ell_0 = |y_2 - \ell_0| - \ell_0 = y_2 \right) \frac{\underline{r}_2 - \underline{r}_1}{\|\underline{r}_2 - \underline{r}_1\|}$$

$$= -K y_2 \underline{E}_Y$$

(c) $\underline{F} \cdot \underline{E}_y = 0$ for system

Hence $\underline{G} \cdot \underline{E}_Y$ is conserved

$$\underline{G} \cdot \underline{E}_Y = (m_1 + m_2 + m_3)\dot{y}_1 + (m_2 + m_3)\dot{y}_2 + m_3 \ell \dot{\theta} \cos\theta$$

[Note that $\underline{F} \cdot \underline{E}_x \neq 0$ so $\underline{G} \cdot \underline{E}_x$ is NOT CONSERVED]

(d)
$$\dot{T} = \underline{F}_1 \cdot \underline{v}_1 + \underline{F}_2 \cdot \underline{v}_2 + \underline{F}_3 \cdot \underline{v}_3$$

$$= (\underline{N}_1 + m_1 g \underline{E}_x) \cdot \underline{v}_1 + (\underline{N}_2 + m_2 g \underline{E}_x) \cdot \underline{v}_2 + \underline{F}_s \cdot (\underline{v}_2 - \underline{v}_1) + S\underline{e}_r \cdot (\underline{v}_2 - \underline{v}_3) + \underline{N}_3 \cdot \underline{v}_3 + m_3 g \underline{E}_x \cdot \underline{v}_3$$

$$= 0 + 0 - K y_2 \dot{y}_2 + S\underline{e}_r \cdot (-\ell \dot{\theta} \underline{e}_\theta) + 0 + (m_3 g \underline{E}_x \cdot \dot{\underline{r}}_3)$$

$$= -K y_2 \dot{y}_2 + m_3 g \left(\underline{E}_x \cdot \ell \dot{\theta} \underline{e}_\theta = -\ell \dot{\theta} \sin\theta \right)$$

$$= -\frac{d}{dt}\left(\frac{K}{2} y_2^2 - m_3 g \ell \cos\theta \right)$$

Hence $\frac{d}{dt}\left(T + U = E \right) = 0 \Rightarrow E$ is conserved.

where $U = \frac{K}{2} y_2^2 - m_3 g \ell \cos\theta$

Common Errors

a) Several students missed the $m_3 \ell \dot{\theta} (\dot{y}_1 + \dot{y}_2) \cos\theta$ term in T.

b) The main error was incorrectly computing $\underline{F}_s$

c) Although $\underline{G} \cdot \underline{E}_Y$ is conserved, $\underline{G} \cdot \underline{E}_x$ isn't and so too $\underline{G}$ is not conserved.

d) If $\underline{F}_s$ in (b) is incorrect then establishing U is impossible.