Math 185-1

Final Exam

May 9, 2016

Name: ____

- You will have **175 minutes** to complete the exam. The start time and end time will be signaled by the instructor. Do not open the exam or write anything on the exam, including on this cover sheet, until the exam has begun.
- Complete the following problems. In order to receive full credit, please provide rigorous proofs and show all of your work and justify your answers. Unless stated otherwise, you may use any result proved in class, the text, or in homeworks, but be sure to clearly state the result before using it and to verify that all hypotheses are satisfied.
- This is a closed-book, closed notes exam. No electronic devices, including cellphones, headphones, or calculation aids, will be permitted for any reason.
- The exam and all papers must remain in the testing room at all times. When you are finished, you must hand your exam paper to the instructor. In the case of a fire alarm, leave your exams in the room, face down, before evacuating. Under no circumstances should you take the exam with you.
- If you need extra room for your answers, use the back side of each page. You may also use those back sides as well as the spare blank pages at the end of the exam for scratch work. If you must use extra paper, use only that provided by the instructor; make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Do not remove or detach the formula sheet from the exam.

After reading the above instructions, please sign the following:

On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Signature: ____

Stereographic projection:

$$x = X/(1-Z) \qquad X = 2x/(|z|^2+1)$$
$$y = Y/(1-Z) \qquad Y = 2y/(|z|^2+1)$$
$$Z = (|z|^2-1)/(|z|^2+1).$$

Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Harmonic conjugate:

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt - \int_{x_0}^{x} \frac{\partial u}{\partial y}(s,y_0)ds + C$$
$$v(B) = \int_A^B -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy.$$

Fractional linear transformation:

$$w = f(z) = \frac{z-z_0}{z-z_2}\frac{z_1-z_2}{z_1-z_0}.$$

Mean value property:

$$u(z_0) = \int_0^{2\pi} u(z_0+re^{i\theta})\frac{d\theta}{2\pi}.$$

Cauchy integral formula:

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\partial D} \frac{f(w)}{(w-z)^{m+1}}dw.$$

Power series and Laurent series:

$$a_k = \frac{1}{2\pi i}\oint_{|\zeta-z_0|=r} \frac{f(\zeta)}{(\zeta-z_0)^{k+1}}d\zeta.$$

Residue theorem:

$$\int_{\partial D} f(z)dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}[f(z),z_j].$$

$$\mathrm{Res}[f(z),z_0] = \lim_{z\to z_0}(z-z_0)f(z) \qquad \mathrm{Res}[f(z),z_0] = \lim_{z\to z_0}\frac{d}{dz}(z-z_0)^2 f(z) \qquad \mathrm{Res}\left[\frac{f(z)}{g(z)},z_0\right] = \frac{f(z_0)}{g'(z_0)}$$

Argument principle:

$$\frac{1}{2\pi i}\int_{\partial D}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi}\int_{\partial D} d\arg(f(z)) = N_0 - N_\infty.$$

Winding number:

$$W(\gamma,z) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}dw = \frac{1}{2\pi}\int_\gamma d\arg z.$$

Inverse function theorem:

$$f^{-1}(w) = \frac{1}{2\pi i}\int_{|\zeta-z_0|=\rho}\frac{\zeta f'(\zeta)}{f(\zeta)-w}d\zeta, \qquad |w-w_0|<\delta.$$

1. (10 points) Determine the points $z \in \mathbb{C}$ for which $f(z) = |z|^2$ is differentiable, and determine where $f(z)$ is analytic.

Solution: Consider the difference quotient

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

Using the fact that $|z|^2 = z\bar{z}$, we find that

$$\begin{aligned}
\lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z} &= \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\bar{z} + \overline{\Delta z}) - z\bar{z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\bar{z}\Delta z + z\overline{\Delta z} + \Delta z\overline{\Delta z}}{\Delta z} \\
&= \lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} + \frac{\overline{\Delta z}}{\Delta z} z.
\end{aligned}$$

When $z = 0$, this equals 0, so $f'(0)$ exists. If $z \neq 0$, then letting $\Delta z = \epsilon > 0$, we find that

$$\begin{aligned}
\lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} + \frac{\overline{\Delta z}}{\Delta z} z &= \lim_{\epsilon \to 0} \bar{z} + \epsilon + \frac{\epsilon}{\epsilon} z \\
&= \bar{z} + z.
\end{aligned}$$

On the other hand, letting $\Delta z = i\epsilon$, we find

$$\begin{aligned}
\lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} + \frac{\overline{\Delta z}}{\Delta z} z &= \lim_{\epsilon \to 0} \bar{z} - i\epsilon + \frac{-i\epsilon}{i\epsilon} z \\
&= \bar{z} - z.
\end{aligned}$$

Since the limits in the two directions are unequal, $f'(z)$ does not exists when $z \neq 0$.

Hence, f is differentiable only at $z = 0$, and thus, it is not analytic anywhere.

2. (10 points) Let $f(z)$ be a fractional linear transformation such that $f(1) = 2+i$, $f(-1) = i$, and $f(i) = -1+i$. Determine the image of the disk $\{z : |z| < 1\}$ under f.

Solution: Since fractional linear transformations map circle and lines to either circles or lines, the unit circle $|z| = 1$ is sent to either a line or a circle.

As $1, -1$, and i are points on the unit circle, their images $2+i$, i, and $-1+i$ are also in the image of the unit circle. The three points determine a line $y = 1$ in the codomain, so the image of the circle is a line.

The line divides the plane into two regions, and we must determine which is the image of the disk $|z| < 1$. We note that as we traverse the circle counter-clockwise from 1, we first pass through i and then -1, and the disk $|z| < 1$ lies to the left of the curve.

Since fractional linear transformations are conformal, it follows that as we traverse from $f(1) = 2+i$ to $f(i) = -1+i$ and then $f(-1) = i$, the image of the disk must also lie of the left. Thus, the image of the disk $|z| < 1$ is the half-plane $y > 1$.

3. (10 points) Let $u(z)$ be a real-valued harmonic function on a domain D such that $u(z) \geq m$ for all $z \in D$. Show that if $u(z_0) = m$ for some $z_0 \in D$, then $u(z) = m$ for all $z \in D$.

Solution: Since $\Delta(-u) = -\Delta u = 0$, it follows that $-u(z)$ is harmonic. The assumption that $u(z) \geq m$ implies that $-u(z) \leq -m$ for all $z \in D$. The maximum principle then implies that if $u(z_0) = m$ for some $z_0 \in D$, then $-u(z_0) = -m$, so that $-u(z) = -m$ for all $z \in D$.

Hence $u(z) = m$.

4. (10 points) Show that $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

Solution: We know that for $x \in \mathbb{R}$, $\cos^2 x + \sin^2 x = 1$. Since $\mathbb{R}$ is a subset of $\mathbb{C}$ with a non-isolated point, $f(z) = \cos^2 z + \sin^2 z$ and $g(z) = 1$ are analytic on $\mathbb{C}$, and $f(x) = g(x)$ for $x \in \mathbb{R}$, the Uniqueness Principle implies that $f(z) = g(z)$ for all $z \in \mathbb{C}$.

Hence, $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.

5. (10 points) Suppose that $f(z)$ is an analytic function that is surjective onto $\mathbb{C}$ and $g(z)$ has an essential singularity at z_0. Show that $f \circ g$ has an essential singularity at z_0.

Solution: First, note that since z_0 is an isolated singularity of $g(z)$, there exists a $\delta > 0$ such that g is analytic for $0 < |z - z_0| < \delta$. As $f(z)$ is analytic, then $f \circ g$ is analytic for $0 < |z - z_0| < \delta$. Hence, $f \circ g$ has an isolated singularity at z_0.

Recall that z_0 is an essential singularity of a function F if and only if for every $w \in \mathbb{C}$, there exists a sequence $z_n \to z_0$ such that $F(z_n) \to w$.

Let $w \in \mathbb{C}$. By surjectivity of f, there exists a $v \in \mathbb{C}$ such that $f(v) = w$. By the fact that z_0 is an essential singularity of g, there exists a sequence $z_n \to z_0$ such that $g(z_n) \to v$. Since f is analytic, and hence, continuous, we have that

$$\lim_{n\to\infty} f(g(z_n)) = f(\lim_{n\to\infty} g(z_n)) = f(v) = w.$$

Hence, z_0 is an essential singularity of $f \circ g$.

6. (10 points) Compute

$$\int_0^\infty \frac{(\sqrt{x})^{-1}}{1+x}dx,$$

by integrating around a keyhole (i.e. "Pacman") contour.

Solution: We first define a branch of the square root function that is analytic on the interior of the keyhole domain: Let $z \mapsto \sqrt{z}$ be defined by the map $z \mapsto |z|^{1/2}e^{i\,\mathrm{Arg}_1(z)/2}$, where $\mathrm{Arg}_1(z)$ is the branch of the argument such that $0 \le \mathrm{Arg}_1(z) < 2\pi$.

Using this branch of the square root, define $f(z) = \frac{\sqrt{z}^{-1}}{1+z}$. Then $f(z)$ is analytic on the interior of the keyhole contour except a simple pole at $z = -1$. We have that $\mathrm{Res}[f(z), -1] = \frac{\sqrt{-1}^{-1}}{1} = \frac{1}{i}$ by using Rule 3.

Therefore, $\int_\gamma f(z)dz = 2\pi i \frac{1}{i} = 2\pi$ by the Residue Theorem, where γ is the keyhole contour.

Now, let γ_R and γ_ϵ be the circular arcs centered about the origin of radius R and ϵ, respectively. By the ML-estimate, we have that since $|\sqrt{z}| = \sqrt{R}$ for $z \in \gamma_R$ and $|1+z| \ge R-1$ for $z \in \gamma_R$, then

$$\left|\int_{\gamma_R} f(z)dz\right| \le \frac{1}{\sqrt{R}(R-1)}2\pi R.$$

This goes to 0 as $R \to \infty$. Similarly, $|f(z)| \le \frac{1}{\sqrt{\epsilon}(1-\epsilon)}$ on γ_ϵ, so the ML estimate implies that

$$|\int_{\gamma_\epsilon} f(z)dz| \le \frac{1}{\sqrt{\epsilon}(1-\epsilon)}2\pi\epsilon$$

which tends to 0 as $\epsilon \to 0$.

Thus, if we take the limit as $R \to \infty$ and $\epsilon \to 0$, we have that

$$\begin{aligned} 2\pi &= \lim \int_\gamma f(z)dz \\ &= \int_0^\infty \frac{\sqrt{x}^{-1}}{1+x}dx + \int_\infty^0 \frac{\sqrt{x}^{-1}e^{-2\pi i/2}}{1+x}dx \\ &= 2\int_0^\infty \frac{\sqrt{x}^{-1}}{1+x}dx. \end{aligned}$$

Therefore, $\int_0^\infty \frac{\sqrt{x}^{-1}}{1+x}dx = \pi$.

7. (10 points) Determine the number of roots of $z^4 - 3z^3 + 1$ in the right half plane $\operatorname{Re} z > 0$.

Solution: Let D be the half-disk of radius R in the right half plane. We will use the argument principle on $p(z) = z^4 - 3z^3 + 1$ on the domain D, with R very large, to count the number of roots of the function.

Let γ_R be the circular arc of radius R from $-iR$ to iR. On γ_R, we have that if R is sufficiently large, $|z^4| >> | - 3z^3 + 1$, so that $p(z) \approx z^4$. Then,

$$\int_{\gamma_R} d \arg p(z) \approx \int_{\gamma_R} d \arg z^4 = \int_{-\pi/2}^{\pi/2} d \arg(R^4 e^{i4\theta}) = 4\pi.$$

Now, we need to compute the change in argument as we traverse along the straight line segment from iR to $-iR$. Note that $p(iy) = y^4 + 1 + 3iy^3$. So when $R >> 1$, we have that $p(iR) \approx R^4 + 3iR^3$, so that $p(iR)$ is slightly above the positive real axis. Along the straight line from iR to $-iR$, $p(iy)$ only meets the real axis when $y^3 = 0$, or in other words, at $y = 0$. At $y = 0$, $p(iy) = p(i0) = 1$, so $p(i0)$ is on the positive real axis. Thus, the change in argument of $p(z)$ from $p(iR)$ to $p(0)$ is approximately 0.

Similarly, we have that $p(-iR) \approx R^4 - 3iR^3$, so $p(-iR)$ is just below the positive real axis. As $p(i0)$ is on the positive real axis and $p(iy)$ does not cross the real axis again between $p(i0)$ and $p(-iR)$, the change in argument is approximately.

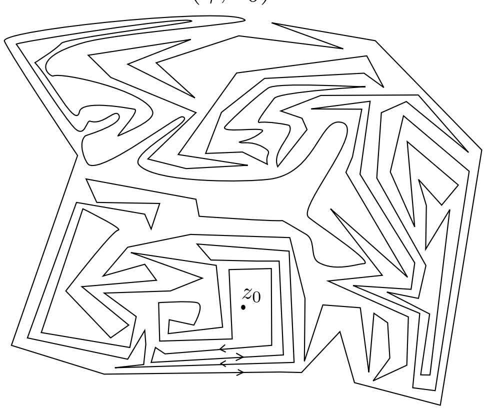
Hence, $\int_{\partial D} d \arg p(z) \approx 4\pi + 0 + 0$.

By the argument principle the integral above is an integer multiple of 2π, so must equal 4π. Moreover, it is equal to $2\pi(N_0 - N_\infty)$, where N_0 is the number of zeros in D and N_∞ is the number of poles of $p(z) in D$. As $p(z)$ is analytic, $N_\infty = 0$, so $N_0 = 2$, and $p(z)$ has 2 zeros in the right half plane.

8. (a) (5 points) Show that the winding number is locally constant – that is if γ is a piecewise smooth closed curve and $z_0 \in \mathbb{C} \setminus \gamma$, there exists a $\delta > 0$ such that $W(\gamma, z) = W(\gamma, z_0)$ for all $|z - z_0| < \delta$.

Solution: Let $W(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{f'(w)}{f(w)-z} dw$. Then, $W(\gamma, z)$ is an analytic function in z for $z \notin \gamma$, so in particular, it is continuous. Fixing z_0 and letting $\epsilon = 1/2$, continuity at z_0 implies that there exists $\delta > 0$ such that for $|z - z_0| < \delta$, we have that $|W(\gamma, z) - W(\gamma, z_0)| < 1/2$. But as $W(\gamma, z)$ is always an integer, it follows that $W(\gamma, z) = W(\gamma, z_0)$ for $|z - z_0| < \delta$.

(b) (5 points) Let γ be the curve illustrated below with the point z_0 as indicated. Determine $W(\gamma, z_0)$.



Solution: The curve γ is simple, so by the Jordan curve theorem, $W(\gamma, z_0)$ is equal to 0 or ± 1, with the winding number being 0 if it is in the unbounded component and ± 1 if it is in the component bounded by γ.

Take a point z below z_0 on the unbounded component outside the picture of the curve. Draw a straight line between z and z_0. By the Jump Theorem for the winding number, we know that the winding number changes by ± 1 each time we cross γ. The straight line from z to z_0 cross γ four times, so this implies that $W(\gamma, z_0)$ must be even, so equal to 0.

9. (a) (5 points) Suppose that $\{f_n(z)\}_{n=1}^{\infty}$ is a sequence of analytic functions on a domain D that converges uniformly to f on D. Show that $f(z)$ is analytic on D.

Solution: Fix $z_0 \in D$. There exists an open ball $|z - z_0| < \delta$ contained inside D since D is open. Since $f_n(z)$ is analytic on D, the Cauchy integral formula implies that for $z \in D$,

$$f_n(z) = \frac{1}{2\pi i} \int_{|w-z_0|=\delta} \frac{f_n(w)}{w-z} dw.$$

Since $f_n(z) \to f$ uniformly, it follows that if $|z - z_0| < \delta/2$, then $f_n(w)/(w-z)$ converges uniformly to $f(w)/(w-z)$ on $|z - z_0| = \delta$, as the denominator has modulus bounded below by $\delta/2$ for all $|w - z| = \delta$. Thus, if $|f_n(w) - f(w)| < \epsilon_n$ where $\epsilon_n \to 0$, then $|f_n(w)/(w-z) - f(w)/(w-z)| < \frac{\epsilon_n}{\delta/2}$ and $\frac{\epsilon_n}{\delta/2} \to 0$.

Hence, the integrands converge uniformly on $|w - z| = \delta$, so

$$f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{|w-z_0|=\delta} \frac{f_n(w)}{w-z} dw = \frac{1}{2\pi i} \int_{|w-z_0|=\delta} \frac{f(w)}{w-z} dw.$$

From this expression and an exercise from the homework (Exercise III.1.6), we see that $f(z)$ is differentiable on $|z - z_0| < \delta/2$, so in particular, it is analytic at z_0. Since this holds for any arbitrary $z_0 \in D$, then f is analytic on D.

(b) (5 points) Suppose also that $f_n(D) \subseteq D$ for all n. Show that $f(D) \subseteq D$. (Hint: For $z \in D$, first conclude that $\lim\limits_{n\to\infty} f_n(z) \in \bar{D} = D \cup \partial D$. Then apply the Open Mapping Theorem.)

Solution: Fix $z \in D$. Since f_n converges uniformly on D, it follows that $\lim_{n\to\infty} f_n(z)$ exists. As $\bar{D} = D \cup \partial D$ is the set of limit points of D and $f_n(z) \in D$ for all n, it follows that $f(z) = \lim_{n\to\infty} f_n(z) \in \bar{D}$.

To show that $f(z) \in D$, it suffices to show that $f(z) \notin \partial D$. Recall that ∂D consists of points such that every ϵ-ball around the point intersects both D and D^C. But since f is analytic by part (a), the Open Mapping Theorem says that $f(D)$ is open. Hence, for all $z \in D$, $f(z)$ has an ϵ-ball around $f(z)$ that is wholly contained in $f(D)$. We conclude then that $f(z) \notin \partial D$, so $f(D) \subseteq D$.

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Question:	1	2	3	4	5	6	7	8	9	Total
Points:	10	10	10	10	10	10	10	10	10	90
Score:										