Solutions

1. [25 points] Short Questions

(a) [5 points] Circle T or F for True or False

T **F** (i) The principle of equipartition of energy states that in equilibrium the thermal energy is shared among all active degrees of freedom and is randomly distributed with an average energy of kT/2.

T **F** (ii) If the temperature difference across a conductor triples, the rate at which it transfers heat energy increases by a factor of nine.

T F (iii) When a system goes from equilibrium state 1 to state 2, the change in the internal energy is the same for all processes.

T F (iv) The internal energy of a given amount of an ideal gas at equilibrium depends only on its absolute temperature.

T F (v) For any material that expands when heated, C_P is greater than C_V.

(b) [5 points]

T F (vi) The linkage of flux from circuit A to circuit B is the same as the linkage of flux from circuit B to circuit A.

T F (vii) All materials are diamagnetic. It is sometimes over ridden by ferromagnetism or paramagnetism.

T **F** (viii) Lenz's law states that two parallel wires carrying current in the same direction will oppose each other and push each other apart.

T F (ix) The wave equation can be derived from Maxwell's equations.

T F (x) Electromagnetic waves are transverse waves.

(c) [5 points] A circuit contains a capacitor and resistor in series is connected to an AC source. The circuit schematic is shown in the accompanying figure.

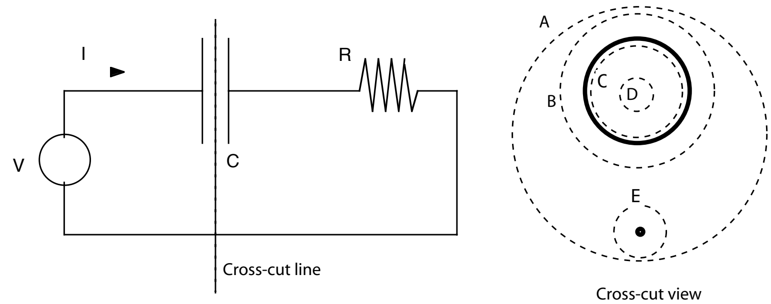


Figure 4: sketch of circuit for problem 1 (c)

The right portion of the figure shows a cross-section through the circuit at the mid-point of the parallel plate capacitor. There are 5 imaginary closed paths (A, B, C, D, E) drawn around portions of the capacitor and the return wire (the wire coming back to the AC source).

(i) Rank the absolute value of the $\int \vec{B} \cdot d\vec{s}$ in order (maximum = 1 to minimum = 5). (Break ties by which has the highest mean absolute or rms magnetic field.)

Path	(i) $\lvert\int \vec{B} \cdot d\vec{s}\rvert$	(ii) $\lvert\int \vec{E} \cdot d\vec{s}\rvert$
A	5	1 (2)
B	1(2)	1 (1)
C	3	3
D	4	4
E	1 (1)	5

(ii) If the capacitor is replaced by a solenoid - inductor that produces a uniform magnetic field inside the solid circle (between imaginary circles C and B and perpendicular to them), rank the absolute value of $\int E \cdot d\vec{s}$ around the imaginary paths in order (maximum = 1 to minimum =5).

(d) [5 points] Circle correct answer

(i) If the rms voltage in an AC circuit (made of resistors, capacitors, and inductors) is doubled, the peak current is

(A) increased by a factor of 2. from complex Ohm's law $I = V/Z$
(B) decreased by a factor of 2.
(C) increased by a factor of $\sqrt{2}$.
(D) decreased by a factor of $\sqrt{2}$.
(E) not enough information to determine the change.

(ii) If the current in an inductor is doubled, its stored energy will

(A) increase by a factor of 2.
(B) decrease by a factor of 2.
(C) increase by a factor of 4. $U = \frac{1}{2}LI^2$
(D) increase by a factor of 8.
(E) not changed.

(iii) A positively charge particle is moving northward in a static magnetic field The magnetic force on the particle is toward the northeast. What is the direction of the magnetic field?

(A) Upward
(B) West
(C) South
(D) Downward
(E) This situation cannot exist. $F_M = q\vec{v} \times \vec{B}$ means $\vec{F} \perp \vec{v}$
(F) Upward at an angle of 45° to East .

(iv) If the AC frequency driving an inductor is doubled, the inductive reactance of the inductor will

(A) increase by a factor of 2. $|X_L| = \omega L$
(B) not change.
(C) decrease by a factor of 2.
(D) increase by a factor of 4.
(E) decrease by a factor of 4.

(v) An ideal transformer has N_p turns on the primary and N_s turns on its secondary. The power dissipated in a load resistance R connected across the secondary is P_s, when the primary voltage is V_p. The current in the primary is then

(A) P_s/V_p
(B) $\frac{N_p}{N_s}\frac{P_s}{V_p}$
(C) $\frac{N_s}{N_p}\frac{P_s}{V_p}$
(D) $\left(\frac{N_s}{N_p}\right)^2 \frac{P_s}{V_p}$
(E) $\left(\frac{N_s}{N_p}\right)^2 \frac{V_p^2}{R}$ also correct

(e) [5 points] Circle correct answer

(i) Compasses point north because

(A) the north star attracts them.
(B) the Earth has an electric charge.
(C) there are electric currents in the iron core of the Earth.
(D) there are magnetic monopoles near the North Pole.
(E) there is a very large bar magnet in the Earth.

(ii) The Earth's magnetic field flips are used for

(A) creating new permanent magnets.
(B) proving the Earth has a solid iron core..
(C) generate useful power.
(D) geologic dating.
(E) bird migration.

(iii) Which is **not** a property of EM waves?

(A) E and B waves have the same velocity $v_E = V_B = c = 1/\sqrt{\epsilon_0 \mu_0}$.
(B) E and B waves are in phase.
(C) EM waves are transverse with E and B perpendicular to the direction of wave motion
(D) E and B magnitudes are in the ratio $E/B = c$.
(E) EM waves can self-propagate. They require no medium for propagation.
(F) EM waves carry both energy and momentum.
(G) The electric field of a wave decreases as one over the square of the distance from the source.

(iv) Magnetism comes from

(A) magnetic monopoles.
(B) moving quanta of light.
(C) quantization of charge.
(D) moving electric charge.
(E) magnesia and lodestones transferred to soft iron.

(v) AC is used instead of DC power because

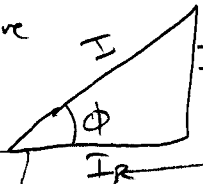
(A) it is safer.
(B) it carries more power.
(C) it makes the use of transformers straight forward.
(D) it is higher voltage
(E) Westinghouse had deeper-pocket backers than Edison.

2. [25 points] A resistor R, induct or L, and a capacitor C are connected in parallel to an AC voltage source V at angular frequency ω.

(a) [5 points] Find the current I and RMS current I_{RMS} from source V

First we must draw the phasor diagram. (see part c)

Then by right triangles we have



so $I = \sqrt{I_R^2 + (I_c - I_L)^2}$

so $\boxed{I = V\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}}$ $\boxed{I_{rms} = \frac{I}{\sqrt{2}}}$

$\frac{1}{2}$ pt for $I_{rms} = \frac{I}{\sqrt{2}}$
4.5 pt for getting I somehow

Wrong ways to get I (out of 4.5)
use series formulae (+0.5)
add in parallel wrongly: $\frac{1}{Z} = \frac{1}{X_R} + \frac{1}{X_L} + \frac{1}{X_C}$ (+2.5)
$\frac{1}{Z} = \sqrt{\left(\frac{1}{X_R}\right)^2 + \left(\frac{1}{X_L} + \frac{1}{X_C}\right)^2}$ (+3.5)
set $I_R = 0$ but otherwise ok (+1.5)

$-\frac{1}{2}$ for messing up impedances or making nontrivial sign goofs

(b) [5 points] Find the phase angle ϕ between V and I.

$\tan\phi = \frac{I_c - I_L}{I_R} = \frac{\omega C - \frac{1}{\omega L}}{1/R} = \omega RC - \frac{R}{\omega L}$

Thus $I(t) = V\sqrt{\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2}\,\sin(\omega t + \phi)$

for voltage source $V(t) = V\sin(\omega t)$

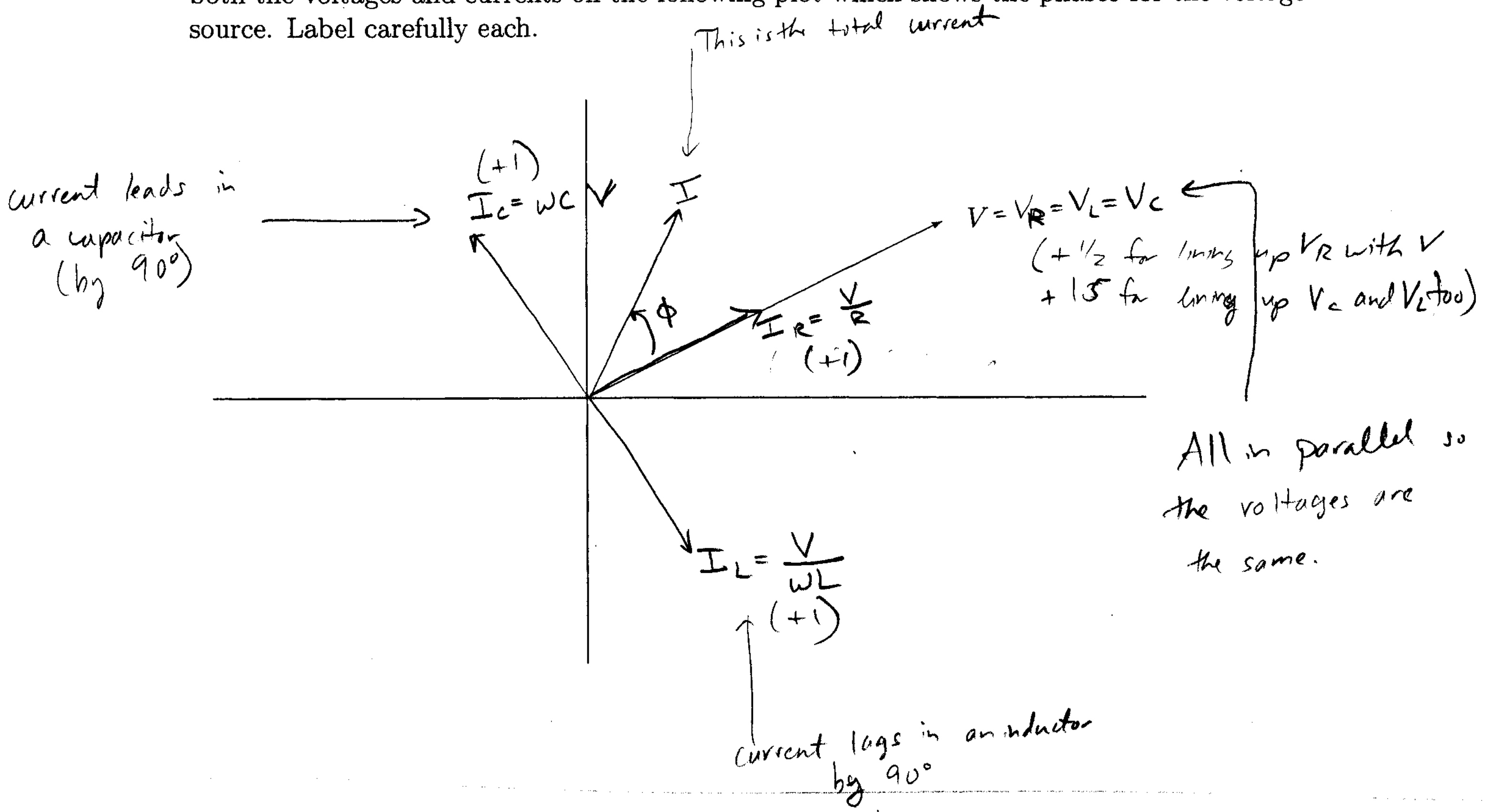
Grading scheme for part b:
no credit for
$P = I^2 Z\cos\phi$.

$\cos\phi = \frac{R}{Z}$ or $\tan\phi = \frac{X_L - X_C}{R}$ (1 pt)
$\tan\phi = \frac{\omega L - \omega C}{R}$ ($\frac{1}{2}$ pt)
$\tan\phi = \sqrt{I_R^2 + (I_c - I_L)^2}$ (3 pts)
start with correct $\tan\phi$ then mess up (4 pts)

→ trig approach (writing in terms of $\sin\omega t$ & $\cos\omega t$)
+1 for setting $V = X_p I_p$ for $p = I, C, L$ (each branch gets same voltage)
+1.5 for setting $I = I_R + I_L + I_C$ as amplitudes
or
+2.5 for setting $I(t) = I_R(t) + I_L(t) + I_C(t)$ as functions of time

+1 for taking $I(t) = I_R\sin\omega t + (I_c - I_L)\cos\omega t$ and writing as $I(t) = I\sin(\omega t + \phi)$

(c) [5 points] Make a diagram showing the phasors for each of the three components. Show both the voltages and currents on the following plot which shows the phasor for the voltage source. Label carefully each.



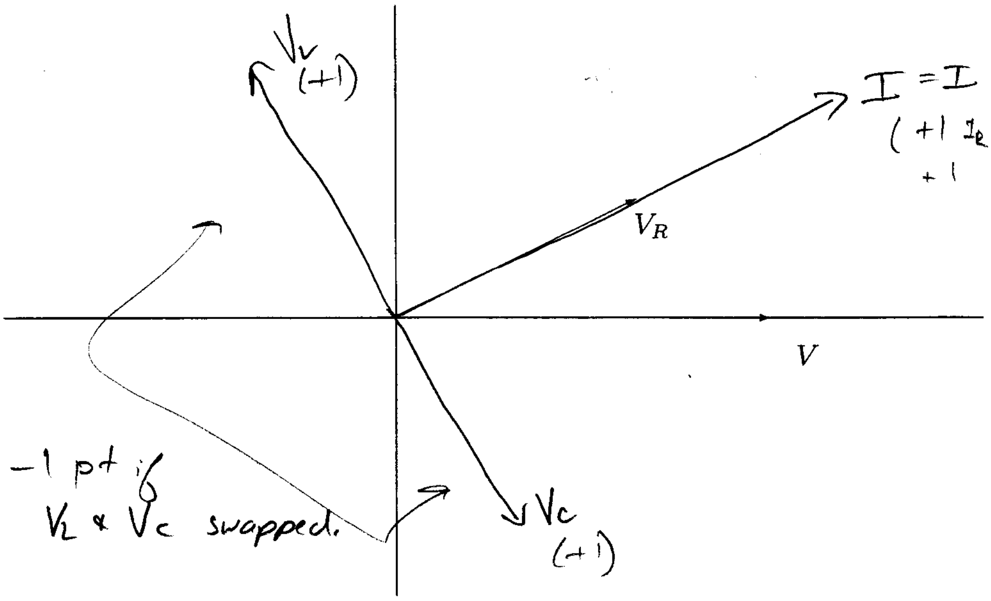
Grading scheme

- lining up V_R with V (+½)
- lining up V_L & V_C with V and/or V_R (+1.5)
- lining up I_R with V_R (+1)
- Placing I_C + I_L correctly w/respect to V_C + V_L (+1 each)

Specific mistakes: did pure series picture (0 pts)
swap I_C + I_L (−1 pt)
implicitly assume $V = V_R = V_L = V_C$ but nowhere is that stated (−1)
pure series but line up V_R with V (+½ pt)

(d) [5 points] If the components, R, L, and C are connected in series with the AC voltage source, draw the phasor diagram showing currents and voltages. Given the voltage phasor of the source V and the voltage phasor of the resistor V_r

series, so same I



(e) [5 points] In the series circuit What value of capacitance C delivers the most power into the resistor R, when $R = 90\Omega$, $L = 100$ microHenries, and $\omega = 10^5$ Hz? What is that power?

$$P = I^2 R = \frac{V^2 R}{Z^2} = \frac{V^2 R}{R^2 + (\omega L - \frac{1}{\omega C})^2}$$ (+1)

(or $P = I^2 Z \cos\phi = I^2 R$)

minimize Z^2 to maximize P

Z^2 minimum when $Z^2 = R^2$ ⇒ $\omega^2 = \frac{1}{LC}$ ⇒ $C = \frac{1}{\omega^2 L} = 10^{-6}$ F (+2.5) (+0.5)

so $P = \frac{V^2}{R}$ (+1)

Specific cases: $P = I^2 Z$ (+1/2 instead of +1)

try to do "impedance matching" $R = X_L + X_C$ (+1/2 pt)

declare P_{max} when $C = \infty$ (+1 to give at most 2 out of 5)

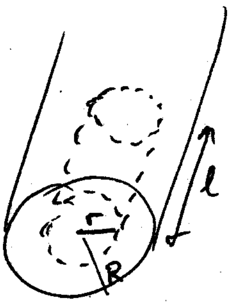
try to minimize Z and fail (+2)

get really close to correct answer except need $P = I_0^2 R$ (+2)

Smoot #3

Brad Hagan

[4] (a)



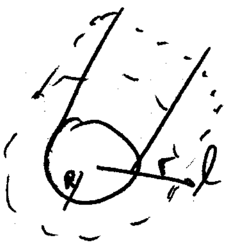
Gauss' Law

LHS: 2 points RHS: 2 points

$$\oint E\,dA = E\cdot 2\pi r l = \frac{\rho\cdot\pi r^2 l}{\varepsilon_0}$$

$$E = \frac{\rho r}{2\varepsilon_0}$$

[4] (b)



LHS: 2 points RHS: 2points

$$\oint E\cdot dA = E\cdot 2\pi r l = \frac{\rho\pi R^2 l}{\varepsilon_0}$$

$$E = \frac{\rho R^2}{2\varepsilon_0 r}$$

Note: 0 points for $\vec{E} = \int \frac{k\,dq\,\hat{r}}{r^2}$ unless I felt there was some knowledge shown

[5] (c) Take $V=0$ at $r=0$

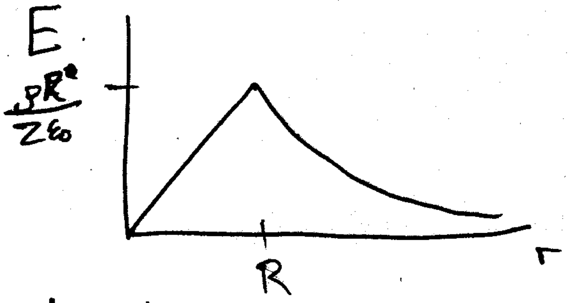
inside: $V(r) - V(0)^{0} = -\int_0^r \frac{\rho r'}{2\varepsilon_0}dr' \Rightarrow \boxed{V(r) = -\frac{\rho r^2}{4\varepsilon_0}}$ inside

outside: $V(r) - V(R) = -\int_R^r \frac{\rho R^2}{2\varepsilon_0 r'}dr'$

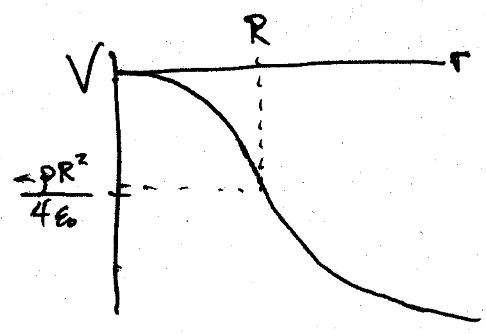
$$V(r) + \left(\frac{\rho R^2}{4\varepsilon_0}\right) = -\frac{\rho R^2}{2\varepsilon_0}\ln\left(\frac{r}{R}\right) \Rightarrow V(r) = -\frac{\rho R^2}{2\varepsilon_0}\left(\frac{1}{2} + \ln\left(\frac{r}{R}\right)\right)$$

Grading: 1 pt each for integrating E in each region.
1 pt for a valid 0 point ($V(\infty)=0$ not valid)
2 pts for matching the V at surface and keeping your 0 points consistent.

[2] (d)



1 point



1 point

Notes: I wanted the correct shapes. I was not grading for consistency w/ parts (a)–(c).

Solution to Problem 4: Hybrid Car

(a) The electric motor will be made up of multiple loops of current rotating in the permanent magnetic field of 1 Tesla. The torque exerted on a current loop in an external magnetic field is

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

Considering all the turns as separate current loops, the total torque will just be N times this value, or

$$\vec{\tau} = N(I\vec{A} \times \vec{B})$$
$$\vec{\tau} = NIAB\sin\alpha\hat{\tau}$$

where α is the angle between the orientation of the magnetic field and the current loop. The maximum torque occurs when the two are at an angle of 90 degrees to one another, or when $\sin\alpha = 1$. Thus,

$$\tau_{max} = NAIB = \frac{NAVB}{R} = \frac{(0.3m^2)(200V)(1T)}{(0.3\Omega)} = 200N\ m = 147.49\,ft\ lb$$

(b) As the current loop rotates faster (more RPM) the flux through the current loop is changing more rapidly, and this creates a bigger back-emf. This back-emf limits the current supplied to the loop, and thus reduces the torque. So, we should find that torque decreases with increasing RPM. We do the calculation as above, except we make the replacement

$$V \rightarrow V - emf$$

where *emf* is the back-emf. What is this value? From Faraday's Law,

$$emf = N\frac{d\Phi_B}{dt} = N\frac{d}{dt}(\vec{A}\ \vec{B}) = N\frac{d}{dt}(AB\cos\alpha)$$

If the loop is rotating with angular frequency ω then we can say $\alpha = \omega t$ and

$$emf = N\frac{d}{dt}(AB\cos\omega t) = NAB\omega\sin\omega t = \frac{2\pi}{60}NABf\sin\omega t$$

where *f* is the RPM. Thus, the torque is now

$$\tau = \frac{NA(V - emf)B}{R}\sin\omega t = \frac{NA(V - \frac{2\pi}{60}NABf\sin\omega t)B}{R}\sin\omega t$$

This will be at a maximum when $\sin\omega t = 1$, so

$$\tau_{max} = \frac{NAVB}{R}\left(1 - \frac{2\pi}{60}NABf\right) = (200N\ m)\left(1 - \frac{2\pi}{60}(0.3m^2)(1T)\,f\right) = \left(200 - .01\pi f\right)N\ m$$

or

$$\tau_{max} = (147.5 - 0.0232\,f)\,ft\ lb$$

This is a maximum of 147.5 ft-lb at 0 RPM and drops to 0 ft-lb at 6,367 RPM.

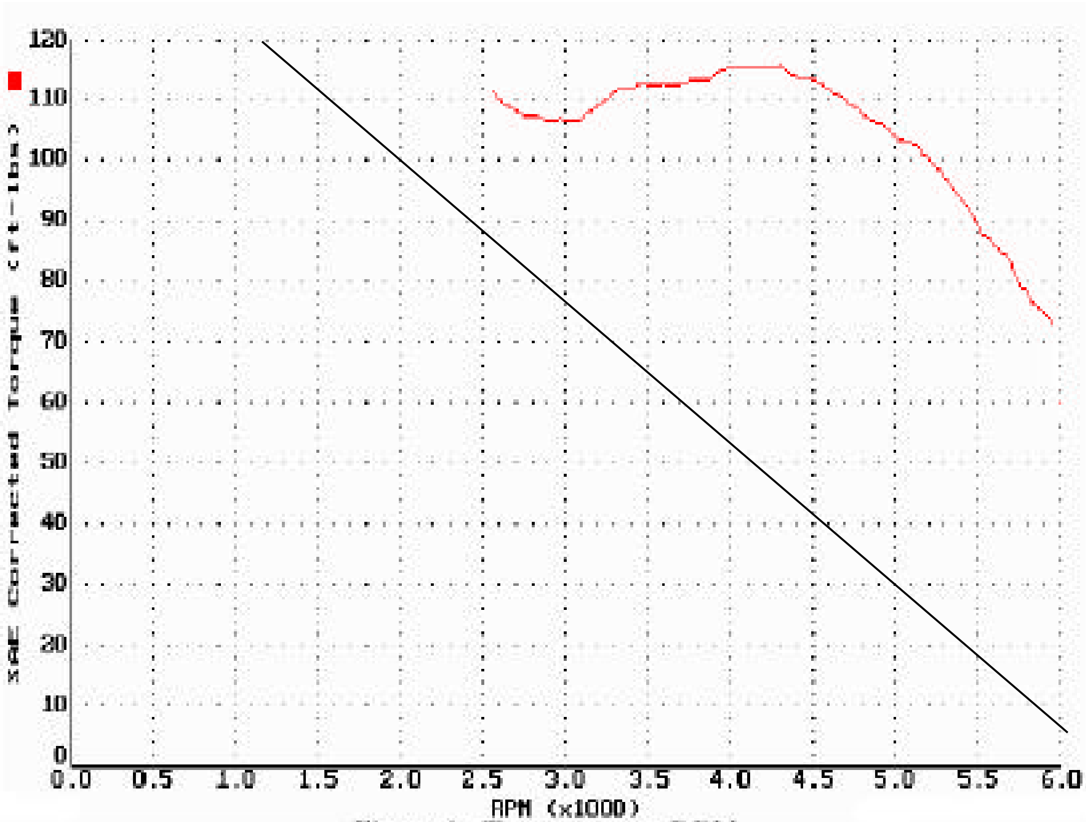


Figure 2: Torque versus RPM

(c) Using the first hint, and our calculated torque from part (b), we get

$$P = \tau\omega = \left[(200 - .01\pi f)N \cdot m\right]\frac{2\pi}{60}f = (20.94f - .003289f^2)W = (.02807f - .000004411f^2)hp$$

This is a parabola. The power is 0 at 0 RPM and 6,367 RPM, and reaches a maximum of 44.6 hp at 3,184 RPM.

Using the second hint, we get

$$P = emf \cdot I = emf\,\frac{V - emf}{R}$$

Using our expression for *emf* found above, we get

$$P = \frac{2\pi}{60}NABf\,\frac{V - \frac{2\pi}{60}NABf}{R} = (.02807f - .000004411f^2)hp$$

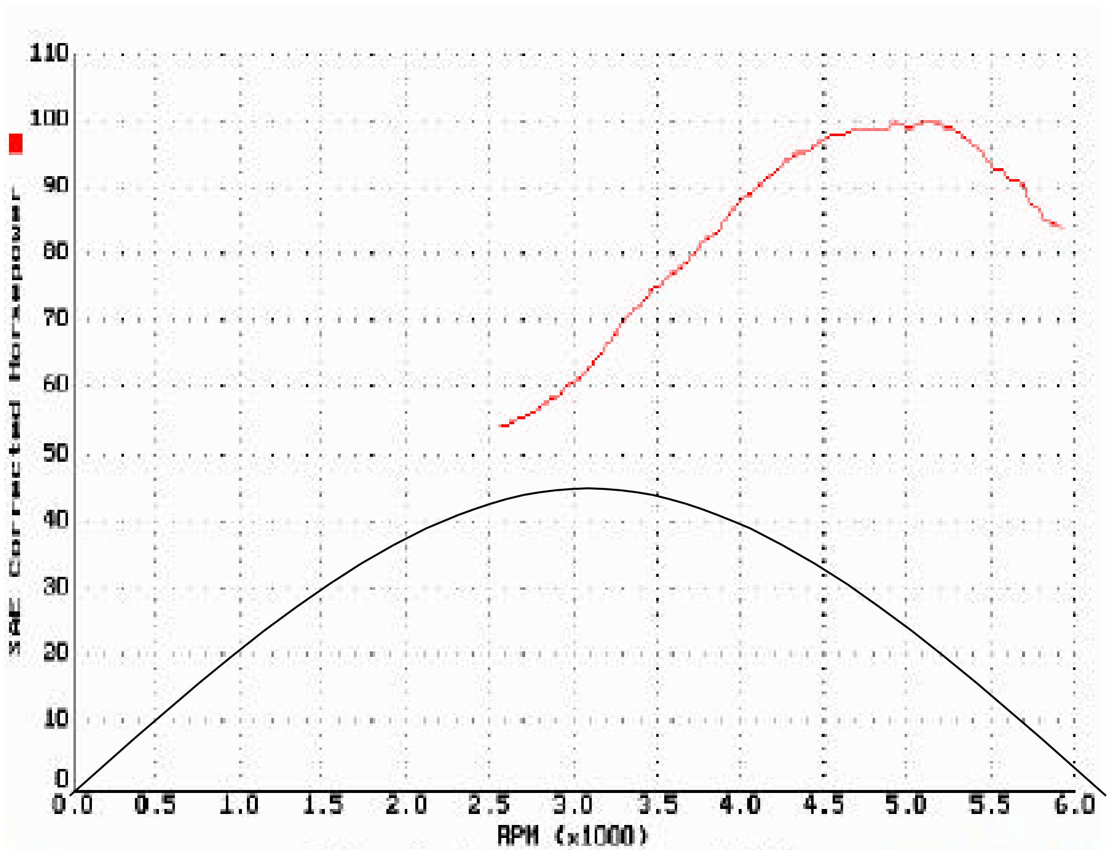


Figure 3: Horse Power versus RPM

(d) The hybrid uses 1 gallon of gasoline in an hour at 60 mph at 60 mpg. All the energy in the gasoline gets turned into heat energy, so the heat energy produced is just the energy stored in a gallon of gasoline, 60kW-hr, or

$$E = 60{,}000\,W \; hr \frac{60\min}{1hr}\frac{60\,sec}{1\min} = 2.16\times10^{8}\,J$$

The total mass of gasoline is

$$\text{total mass} = 2.16\times10^{8}\,J \frac{1kg}{4.56\times10^{7}\,J} = 4.73kg$$

Since the ratio of carbon atoms to hydrogen atoms is 8 to 18, the percentage of the total mass that is carbon is

$$\frac{mass_C}{mass_{Total}} = \frac{mass_C}{mass_C + mass_H} = \frac{8\cdot 12}{8\cdot 12 + 18\cdot 1} = \frac{96}{114} = .842$$

Thus the mass of carbon is

$$mass_C = .842\,mass_{Total} = 3.98kg$$

$$mass_H = .158\,mass_{Total} = 0.75kg$$

And so the masses of the products are

$$mass_{C0_2} = \frac{12+16+16}{12} mass_C = \frac{44}{12} \; 3.98kg = 14.59kg$$

$$mass_{H_2O} = \frac{1+1+16}{2} mass_H = \frac{18}{2} \; 0.75kg = 6.66kg$$

1 mole of ideal gas at STP occupies 22.4L. Therefore, we need only find the number of moles of each gas to calculate the volume they occupy:

$$V_{CO_2} = (22.4L)n_{CO_2} = (22.4L) \; 14.59kg \left(\frac{1mol}{.044kg}\right) = 7{,}428L$$

$$V_{H_2O} = (22.4L)n_{H_2O} = (22.4L) \; 6.66kg \left(\frac{1mol}{.018kg}\right) = 8{,}288L$$

For a total volume of 15,716L.

Smoot #5

Brad Hagan

[4] (a) $\lambda = \frac{c}{f}$ Old range: 0.336m – 0.361m 2 points
PCS: 0.158m, 1 point; Microwave: 0.122m 1 point

–1 point if no units.

[4] (b) $\bar{S} = \frac{P}{A} = \frac{500\text{ W}}{\frac{1}{30}\cdot(4\pi[5000m]^2)} = 4.78\times10^{-5}\text{ W/m}^2$ ← 2 points for getting $\bar{S}$

$= \frac{E_{rms}^2}{\mu_0 c} \Longrightarrow E_{rms} = \sqrt{\mu_0 c \bar{S}}$ ← 2 points for relating E and $\bar{S}$

$= 0.13\text{ V/m}$

Also accepted $E_{max} = 0.19\text{ V/m}$

(c) rod: $\mathcal{E} = E\cdot d$

$= (0.13\text{ V/m})(0.15\text{ m}) = 0.0195\text{ V}$

loop: $|\mathcal{E}| = \frac{d\phi}{dt} = NA\frac{dB}{dt} = NA\,\omega\,B_{rms}$ (only want peak $\mathcal{E}$)

$= NA\left(\frac{f}{2\pi}\right)\left(\frac{E_{rms}}{c}\right) = 0.116\text{ V}$

rod: 3 points
loop: 4 points
holistic for what fraction of these points you get.

Solution

a) The solution to this part of the problem was much easier than many people made it. Since the current spreads out uniformly in all direction the magnitude of the current density $\vec{J}$ inside the ground is simply given by:

$$\vec{J}(r) = \frac{I}{A}\hat{r} = \frac{100\text{A}}{\frac{1}{2}4\pi r^2}\hat{r}, \tag{1}$$

where the origin of spherical coordinates is at the base of the tower. Above ground the current density is zero.

b) To find the electric field we note that Ohm's law can be written as

$$\vec{J} = \sigma\vec{E}, \tag{2}$$

$$\rho \equiv \frac{1}{\sigma}. \tag{3}$$

So using equations (1),(2) and (3), we have:

$$\vec{E}(r) = \rho\vec{J} = \frac{\rho I}{2\pi r^2}\hat{r} = \frac{1592\text{ V-m}}{r^2}\hat{r}. \tag{4}$$

Evaluated at the man's position, this is:

$$\vec{E}(10\text{m}) = 15.92\text{ V/m }\hat{r} \tag{5}$$

c) To find the voltage as a function of r, we need to perform a line integral of the electric field found in (4). Setting $V = 0$ at $r = \infty$:

$$\begin{aligned} V(r) &= -\int_\infty^r d\vec{r'}\cdot\vec{E}(r') \\ &= -\int_\infty^r dr\frac{\rho I}{2\pi r^2} \\ &= \frac{\rho I}{2\pi r} \qquad (6) \\ &= \frac{1592\text{ V-m}}{r} \qquad (7) \end{aligned}$$

Evaluated at the man's position, this is:

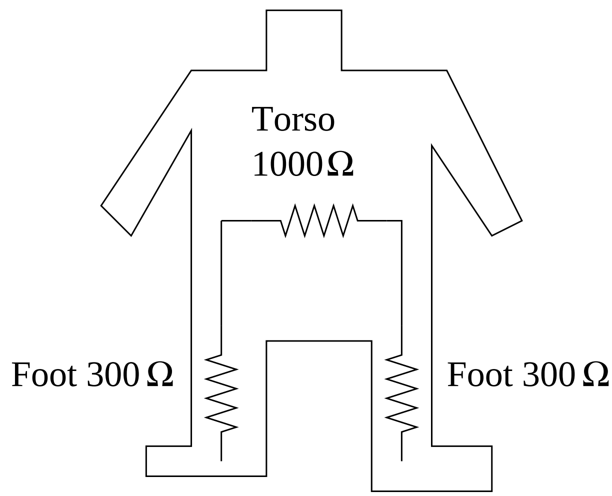
$$V(10\text{m}) = 159.2\text{ V}. \tag{8}$$

It should be noted that since the problem did not specify where the zero of potential was, different answers are possible. Many people set the zero of potential to be the face of the tower.

d) To find the potential differnce between the man's feet, we can evaluate (7) at $r = 10.25$ m and $r = 9.75$ m.

$$\Delta V = 1592 \text{ V-m} \left(\frac{1}{9.75\text{m}} - \frac{1}{10.25\text{m}} \right) = 7.96 \text{ V} \tag{9}$$

e) The circuit diagram for the man looks as below:



So the total current through his torso is:

$$I_{Torso} = \frac{7.96 \text{ V}}{1600\Omega} = 5 \text{ mA} \tag{10}$$

f) The current found in (e) is well below the lethal amount. This man died of "natural" causes.

Grading Scheme

This problem was graded "holistically", meaning that I looked at how the person did the problem and tried to see if they knew what they were doing or not. A lot of people made mistakes on part a) and then had the correct method for the rest of the problem. These people recieved most of the points. Particulary discouraging were those who tried to do this problem using Gauss's law. Some notes as to how the grading was done appear below.

- +2 on part a) for knowing $J = I/A$ but using $A = \pi r^2$ or $4\pi r^2$.

- +1 on part a) $A = \pi r^2/2$
- +3 on part a) for trying to use some form of $R = \rho L/A$ to figure out the resistance of the ground, and using this calculate the current. This was extremely rare.
- -1 on b) for neglecting the fact that the electric field is a *vector* field and failing to indicate the direction of $\vec{E}$, or for having incorrect or no units here.
- Parts c) and d) were often graded together and an aggregate score given for the two parts combined.
- Only 2 points total for c) and d) combined for explaining how to do these parts but not doing them.
- -1 on part c) for obtaining the consistent answer by dubious or suspicious means. A common example of this was having strange bounds on the line integral.
- -2 On e) for only considering the resistance of one foot. This shows that the student really didn't make a mental picture of the current path.
- Additional points were docked if the student provided numerical answers that they should have clearly been able to tell were wrong (e.g. 1×10^7Amps). Many people calculated amounts of energy dissipated in the man on the same order of magnitude as the yearly electrical energy use of the U.S.. "Too Big" or "Too Beacoup" were written where deductions of this kind were made.

Grading scheme for Problem 7: The Earth's Magnetic Field

Overview

The main idea of this problem is that the Earth's magnetic field can be approximated by a current ring around the core of the Earth. Getting the exact numerical answers wasn't too important, but making a reasonable estimate was.

One point (per numerical answer) was taken off for lack of units. One point (per mistake) was taken off for numerical mistakes.

(a) Core Current

Use the Biot-Savart Law (or the formula for magnetic field a distance z above a current loop.)

$$\begin{aligned}
\mathbf{B} &= \int \frac{\mu_0}{4\pi} \frac{I d\mathbf{l} \times \hat{\mathbf{r}}}{r^2} \\
B_z &= \frac{\mu_0}{4\pi} \frac{I \sin\theta}{R_c^2 + R_e^2} \int dr \qquad \sin\theta = \frac{R_c}{(R_c^2 + R_e^2)^{\frac{1}{2}}} \\
B_z &= \frac{\mu_0}{4\pi} \frac{I \sin\theta}{R_c^2 + R_e^2} \cdot 2\pi R_c \\
&= \frac{\mu_0 I R_c^2}{2(R_c^2 + R_e^2)^{3/2}}
\end{aligned}$$

Now we can solve for the current. It was fine if you just wrote down the formula for the magnetic field above a current loop. Recognizing that the problem required Biot-Savart and making a decent but incomplete attempt was awarded up to three points, depending on how good the attempt was.

$$\begin{aligned}
I &= \frac{2(R_c^2 + R_e^2)^{3/2} B_z}{\mu_0 R_c^2} \\
&= 5.04 \times 10^9 \text{ A}
\end{aligned}$$

Note: You absolutely cannot use Ampere's Law to find the current in part (a). The problem does not have the requisite symmetry for Ampere's Law to apply, and in any case, almost no one attempted to draw a loop or define what the enclosed current even meant in this situation. No points were awarded for using Ampere's Law in part (a), or assuming the current was a long straight wire.

(b) Magnetic Moment

$$\begin{aligned}
M &= IA \\
&= I(\pi R_c^2) \\
&= 1.92 \times 10^{23} \text{ A}\cdot\text{m}^2
\end{aligned}$$

Partial credit was given for using the wrong radius: we want the *core* radius because that's where the current is, and the A in the above formula is the area of the current loop itself. Less partial credit was given for using $A = 4\pi R_e^2$.

(c) Energy Stored in Field

If we want to make the approximation $L = \frac{BA}{I}$, then we really ought to use the *central* magnetic field instead of the surface magnetic field. We can see from part (a) (replace R_e with 0) that this central field is given by:

$$B_c = \frac{\mu_0 I}{2R_c}$$

and so

$$L = \frac{\Phi_B}{I} \approx \frac{B_c A}{I} = \frac{B_c(\pi R_c^2)}{I} = \frac{\mu_0 \pi R_c}{2} = 6.87 \text{ H}$$

The problem explicitly asked for L in terms of *geometrical* quantities and asked for a numerical value of L. Both were necessary to get full credit. However, points were not deducted for using incorrect values for I calculated in earlier parts of the problem, or for using the surface magnetic field instead of the central field.

You can't use the formula for energy density stored in the magnetic field to get the total energy because you don't know the magnetic field at any point that's not on the axis of the current loop, and even if you did you'd have to do an impossible integral. However, a small amount of partial credit was awarded for using this method if it was clear that you realized that the method only gets you a rough estimate.

The way to get the energy is using the calculated inductance:

$$U = \frac{1}{2}LI^2 = 8.73 \times 10^{19} \text{ J}$$

(d) Time Constant

$$\tau = \frac{L}{R} \quad \text{for an LR circuit}$$

$$R = \frac{\rho L}{A} = \frac{\rho 2\pi R_c}{\pi R_c^2} = \frac{2\rho}{R_c}$$

$$\tau = \frac{L}{R} = \frac{\mu_0 \pi R_c}{2}\frac{R_c}{2\rho} = \frac{\pi}{4}\mu_0 \sigma R_c^2 = 4.78 \times 10^{12} \text{ s} = 1.52 \times 10^5 \text{ yr}$$

Points were not deducted for using the incorrect L calculated previously. Points were deducted for using R_e instead of R_c. Partial credit was also awarded for attempting to calculate the rate of energy lost by ohmic heating.

(e) Dynamo

We have to modify Ohm's Law because magnetic fields exert a force on moving charged particles. To get full credit, it was necessary to mention the Lorentz force law:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

A lot of answers got tangled up in magnetic fields producing electric fields, which produce magnetic fields, which induce electric fields, which.... but the answer had to do with *forces*. Ohm's Law (in modified form) still works when $\mathbf{E}$ is zero, as in the next part of this question.

Note: The reason we can usually write $\mathbf{E} = \rho\mathbf{J}$ is that the velocity of charged particles is typically very small, so we can ignore that term in Ohm's Law. However, in highly conducting plasmas, $\mathbf{E}$ is small (in a perfectly conducting plasma it would be zero), and so the magnetic force on the particle is important.

If $\mathbf{E} = 0$, and we assume $\mathbf{v}$ and $\mathbf{B}$ are perpendicular, then we obtain:

$$v = \frac{\rho J}{B} \approx \frac{\rho I}{B_c A} = \frac{\rho}{L} = 3.64 \times 10^{-7} \text{ m} \cdot\text{s}^{-1}$$

Significant partial credit was given for finding v and leaving a factor of $\sin\theta$ where θ is the angle between $\mathbf{v}$ and $\mathbf{B}$.

Solution

(a) The heaters are connected in parallel, so they all have the same voltage across them and dissipate the same power. The power dissipated in each resistor is

$$P_1 = \frac{2200 \text{ W}}{8} = 275 \text{ W} \tag{1}$$

Each resistor has 240 rms volts across it, and the power dissipated is given by $P = V_{rms}^2/R$. Solving this for R, we get

$$R = \frac{V_{rms}^2}{P_1} = \frac{(240 \text{ V})^2}{275 \text{ W}} = 210\ \Omega \tag{2}$$

The current is found from $P = I_{rms}V_{rms}$.

$$I_{rms} = \frac{P}{V_{rms}} = \frac{275 \text{ W}}{240 \text{ V}} = 1.15 \text{ A} \tag{3}$$

(b) In the conventional Sauna, heat is transferred first by conduction between the rock and the air, and then by convection to your skin, and then by conduction again into your skin. In the infrared sauna, heat is transferred directly from the heater to you by radiation. The heaters will emit roughly a blackbody spectrum, so the effective area, A_e is found from:

$$P = \sigma A_e T^4 \tag{4}$$

$$\Rightarrow A_e = \frac{P}{\sigma T^4} = \frac{2200 \text{ W}}{(5.67 \times 10^{-8} \text{ W/m}^2\text{-K})\,(200 + 273 \text{ K})^4} = .775 \text{ m}^2 \tag{5}$$

(c) This problem has several parts, all of which are straightforward plug and chug. For (i) We can find the frequency from $f = c/\lambda$, doing this we find a frequency range of:

$$20\ \mu\text{m} > \lambda > 5\ \mu\text{m} \Rightarrow 1.5 \times 10^{13} \text{ Hz} < f < 6 \times 10^{13} \text{ Hz} \tag{6}$$

For (ii), the average electric field in the sauna will be zero, because the sauna will emit a linear combination of plane waves, each of which will have zero electric field on average, so the net average electric field will be zero. The rms field, however will not be zero. The rms field is found from knowing that the time average pointing vector is equal to the power per unit area emitted from the surface, so

$$< S >= \epsilon_0 E_{rms}^2 c = \sigma T^4 \tag{7}$$

$$\Rightarrow E_{rms} = \sqrt{\frac{\sigma T^4}{\epsilon_0 c}} = 1034 \text{ V/m} \tag{8}$$

For (iii), the radiation pressure is

$$P = \frac{< S >}{c} = \frac{\sigma T^4}{c} = 9.46 \times 10^{-6}\ \mathrm{N/m^2} \tag{9}$$

For (iv), the total force on a person, would be the integral of the radiation pressure over their body:

$$F = \int P dA = \int \frac{< S >}{c} dA = \frac{1}{c}(.80)(2200\ \mathrm{W}) = 5.87 \times 10^{-6}\ \mathrm{N} \tag{10}$$

(d) The entropy created in the heater is:

$$\Delta S_{\text{heater}} = \frac{Q}{T} = \frac{(2200\ \mathrm{W})(30 \times 60\mathrm{c})}{200 + 273\ \mathrm{K}} = 8372\ \mathrm{J/K} \tag{11}$$

The entropy change due to the heat transfer from the heater to the people:

$$\Delta S_{\text{transfer}} = -\frac{Q}{T_{\text{heater}}} + \frac{Q}{T_{\text{people}}} = -8372\ \mathrm{J/K} + 12260\ \mathrm{J/K} \tag{12}$$

So adding the results of (**??**) and (**??**), we get

$$\Delta S_{\text{total}} = 12260\ \mathrm{J/K} \tag{13}$$

(e) The temperatures must be converted from Fahrenheight to kelvin.

$$T_H = 550°\mathrm{F} \rightarrow T_H = 561\ \mathrm{K} \tag{14}$$

$$T_L = 50°\mathrm{F} \rightarrow T_L = 283\ \mathrm{K} \tag{15}$$

The Carnot efficiency is then

$$e_{\text{Carnot}} = 1 - \frac{T_L}{T_H} = 1 - \frac{283}{561} = .495 \tag{16}$$

If the efficiency is Carnot, then the engine is reversible, so the entropy change of the universe is zero.

(f) If the efficiency of the engine is η, the work and heat out of the engine are related to the heat in through:

$$W = \eta Q_H \tag{17}$$

$$Q_L = (1 - \eta) Q_H \tag{18}$$

The rate of change of the entropy of the high temperature reservoir is

$$\frac{dS_H}{dt} = -\frac{1}{T_H}\frac{dQ_H}{dt} \tag{19}$$

The rate of change of the entropy of the low temperature reservoir is

$$\frac{dS_L}{dt} = \frac{1}{T_L}\frac{dQ_L}{dt} \tag{20}$$

The rate of change of the entropy of the universe is

$$\frac{dS_T}{dt} = \frac{dS_H}{dt} + \frac{dS_L}{dt} = -\frac{1}{T_H}\frac{dQ_H}{dt} + \frac{1}{T_L}\frac{dQ_L}{dt} \tag{21}$$

Inserting (**??**) and (**??**) into (**??**), using the fact that

$$\eta = .8 e_{\text{Carnot}},$$

and doing some algebra, one eventually arrives at

$$\frac{dS_T}{dt} = \frac{1}{4T_L}\frac{dW}{dt} = \frac{1}{4 \times 283 \text{ K}} 2200 \text{ W} = 1.94 \text{ J/K} \tag{22}$$

Grading Scheme

This problem was pretty straightforward, and was broken into many smaller parts.There was not much partial credit to give here - you either knew what formula to use or you didn't. In general a point was deducted from answers that had the wrong units or no units.

Part (a):

- +1 for thinking that 2200 W was the power of a single heater and that they were supposed to find the total resistance and current.
- +2.5 for the current and +2.5 for the resistance. I was flexible as to whether they put the rms or peak current, as the problem did not state.
- -1 for a silly mistake in calculating the current or the resistance.
- +2 for calculating the *total* resistance and *total* current, but not the current and resistance of one resistor.

Part (b):

- +1 for *each* correct identification of a heat flow process.
- +1 total for the whole identification part if they explained how conduction, convection, and radiation work, but not how they are working in this sauna.
- Some subjective ness was used to determine how many points for the explanations
- +0 for just saying "conduction, convection, and radiation" with no explanation of how they work in the heater.
- +2 for finding the effective area.

Part (c):

- Each part of this problem was worth one point with no partial credit, except for part (ii), which was worth 2 points - one for the average electric field and one for the rms electric field. Some partial credit was given if the person sketched out *exactly* how to do the problem with equations but didn't plug in.
- +.5 for 7.36×10^{-6} N on part (iv), which forgets that only 80% of the energy is going into the person.

Part (d):

- +3 for having the wrong answer because you thought *each* resistor dissipated 2200 W, and you already lost points on (a) for this.
- +1.5 for getting the transfer part right, but forgetting to add the entropy "created" by the heater.

Part (e):

- +0 points for evaluating the Carnot formula using temperatures not on the absolute scale. This indicates a lack of understanding of how this formula works.
- + .5 points for having the correct formula, but evaluating it incorrectly with some attempt to use absolute temperature perhaps from incorrect temperature conversion.
- +1.5 for correctly calculating the efficiency
- +1.5 for realizing that since the efficiency is Carnot, the change in entropy is zero.

Part (f):

- +4 for doing this correctly but thinking that *each* resistor put out 2200 W and that the total power was 8×2200 W.
- +.5 for an approximation like $\Delta S \approx eQ_H/T$ that gets close.