# EECS 16A Designing Information Devices and Systems I
# Spring 2018 Midterm 1

**Midterm 1 Solution**

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ____________________, ____________________ ____________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ________________________________

Name and SID of the person to your left: ________________________________

Name and SID of the person to your right: ________________________________

Name and SID of the person in front of you: ________________________________

Name and SID of the person behind you: ________________________________

**1. What did you do over winter break? (1 Point)**

**2. What activity do you really enjoy? (1 Point)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

**3. System of Equations (4 points)**

Solve the following system of equations using Gaussian elimination. If there is no solution, explain why.

$$x+3y-z=4$$
$$4x-y+2z=8$$
$$2x-7y+4z=-3$$

**Solution:**

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 4 & -1 & 2 & 8 \\ 2 & -7 & 4 & -3 \end{array}\right] \underbrace{\Longrightarrow}_{R_3 \leftarrow -2R_1+R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 4 & -1 & 2 & 8 \\ 0 & -13 & 6 & -11 \end{array}\right]$$

$$\underbrace{\Longrightarrow}_{R_2 \leftarrow -4R_1+R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & -13 & 6 & -11 \end{array}\right]$$

$$\underbrace{\Longrightarrow}_{R_3 \leftarrow R_2-R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 4 \\ 0 & -13 & 6 & -8 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

Final equation effectively means that 0 = 3, which is not possible, and implies that there is no solution to the system of equations.

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**4. Eig-dentical Eigenvalues (10 points)**

(a) (2 points) Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ -1 & a \end{bmatrix}$$

Is $\vec{v} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$ in the column space of $\mathbf{A}$ when $a = 3$? Justify your answer in 1-2 sentences.

**Solution:** When $a = 3$, the two columns of the matrix $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are linearly independent and thus span all of $\mathbb{R}^2$, including $\vec{v}$. You can explicitly solve for the coefficients to obtain the linear combination below, althought this is not necessary.

$$\begin{bmatrix} 3 \\ -8 \end{bmatrix} = \frac{17}{7} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{-13}{7} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

(b) (8 points) Solve for the value of $a$ that yields the **smallest** possible **identical** eigenvalues for the matrix **A**.

**Solution:**

$$det(\mathbf{A} - \mathbf{I}\lambda) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & a-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(a-\lambda) - (-1)(1) = 0$$

$$\lambda^2 - (2+a)\lambda + (2a+1) = 0$$

$$\lambda = \frac{2 + a \pm \sqrt{(2+a)^2 - 4(2a+1)}}{2}$$

Now because we want identical eigenvalues, we know the part under the square root must sum to 0.

$$\therefore (2+a)^2 - 4(2a+1) = 0$$

$$a^2 + 4a + 4 - 8a - 4 = 0$$

$$a^2 - 4a = 0$$

$$a(a-4) = 0$$

So we know that $a = 4$ and $a = 0$ both give us identical eigenvalues.
Now we must choose the $a$ that minimizes the identical eigenvalues. Looking at both cases:

$$a = 4 : \lambda = \frac{2+a}{2} = \frac{2+4}{2} = \frac{6}{2} = 3$$

$$a = 0 : \lambda = \frac{2+a}{2} = \frac{2+0}{2} = \frac{2}{2} = 1$$

$a = 0$ clearly gives us the smaller identical value of the two possibilities and is therefore the solution.

PRINT your name and student ID: ____________________

**5. Nullspace (8 points)**

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 6 \\ 0 & 1 & x \end{bmatrix}$$

(a) (3 points) Find all the values for x such that **M** has a trivial nullspace (this means that the nullspace contains only the zero vector).

**Solution:** There is a trivial nullspace if the matrix is full rank, so we use Gaussian elimination to find the appropriate value(s) of x.

$$\begin{bmatrix} 1 & 0 & 4 \\ 1 & 2 & 6 \\ 0 & 1 & x \end{bmatrix} \underbrace{\Longrightarrow}_{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 1 & x \end{bmatrix}$$

$$\underbrace{\Longrightarrow}_{R_3 \leftarrow 3R_3} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 2 & 2x \end{bmatrix}$$

For the second and third rows to be linearly independent, we need

$$2x \neq 2$$

$$\Rightarrow x \neq 1$$

(b) (5 points) Find a value for $x$ such that it has a nontrivial nullspace and solve for the nullspace.

**Solution:** From part (a), we set $x = 1$ and row reduce to get

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Then we set $z$ as our free variable and express $x$ and $y$ in terms of $z$.
From row 2 we have

$$2y + 2z = 0, \Rightarrow y = -z$$

From row 1 we have

$$x + 4z = 0, \Rightarrow x = -4z$$

Thus we have the vector below is a basis for the nullspace.

$$\begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$$

**6. Vectors and Bases and Spans, Oh My! (6 points)**

$$\mathbf{A} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 5 \end{bmatrix} \right\}$$

$$\mathbf{B} = \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix} \right\}$$

$$\mathbf{C} = \left\{ \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\mathbf{D} = \left\{ \begin{bmatrix} 2 \\ 4.5 \\ 5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \\ 3.75 \end{bmatrix}, \begin{bmatrix} 1.5 \\ 4.5 \\ 1.25 \end{bmatrix} \right\}$$

(a) (2 points) Select from (A), (B), (C), (D) all sets of vectors above that *span* $\mathbb{R}^3$, and write their corresponding letter below. If none span $\mathbb{R}^3$, write "none".

**Solution:** The sets that span $\mathbb{R}^3$ are (B) and (C).

(A) only contains two vectors, which will never be enough to span $\mathbb{R}^3$; the vectors of (D) form a linearly dependent set:

$$\begin{bmatrix} 2 \\ 4.5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0.5 \\ 0 \\ 3.75 \end{bmatrix} - \begin{bmatrix} 1.5 \\ 4.5 \\ 1.25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) (2 points) Select from (A), (B), (C), (D) all sets of vectors above that *form a basis* for $\mathbb{R}^3$, and write their corresponding letter below. If none form a basis for $\mathbb{R}^3$, write "none".

**Solution:** The set that is a basis for $\mathbb{R}^3$ is (C).

Sets of vectors that do not span $\mathbb{R}^3$ cannot form a basis for it, so the answer cannot be (A) or (D).

Additionally, a basis must be composed of linearly independent vectors, so since (B) contains the zero vector, it is not a linearly independent set and cannot be a basis for anything.

(c) (2 points) Can a set that contains the zero vector ever be a basis for $\mathbb{R}^n$? Explain in one or two sentences why or why not.

**Solution:** It cannot, because any set of vectors containing the zero vector is linearly dependent, and all vectors in a basis must be linearly independent.

**7. Proofs! Oh null! (10 points)**

(a) (3 points) Show that if $\vec{x} \in Null(\mathbf{A})$ then $\mathbf{A}\vec{x} + \mathbf{A}^2\vec{x} + \cdots + \mathbf{A}^n\vec{x} = \vec{0}$.

**Solution:**

$$\begin{aligned} \mathbf{A}\vec{x} + \mathbf{A}^2\vec{x} + \cdots + \mathbf{A}^n\vec{x} &= (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1})\mathbf{A}\vec{x} \\ &= (\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^{n-1})\vec{0} \\ &= \vec{0} \end{aligned}$$

(b) (4 points) Suppose we have a square matrix $\mathbf{A}$ with full rank and a matrix $\mathbf{B}$. Show that $Null(\mathbf{AB}) = Null(\mathbf{B})$.

**Solution:** From above, we know that a vector $\vec{x}$ in the nullspace of $\mathbf{B}$ is also in the nullspace of $\mathbf{AB}$. Now we must show that if a vector $\vec{x}$ is in the nullspace of $\mathbf{AB}$, it is also in the nullspace of $\mathbf{B}$. Note that $\mathbf{A}$ is invertible.

$$\begin{aligned} \mathbf{AB}\vec{x} &= \vec{0} \\ \mathbf{A}^{-1}\mathbf{AB}\vec{x} &= \mathbf{A}^{-1}\vec{0} \\ \mathbf{B}\vec{x} &= \vec{0} \end{aligned}$$

Alternative proof:

Since $\mathbf{A}$ is invertible, its nullspace is trivial. This means if $\mathbf{A}\vec{y} = \vec{0}$, then $\vec{y} = \vec{0}$. So if $\mathbf{AB}\vec{x} = \mathbf{A}(\mathbf{B}\vec{x}) = 0$, then $\mathbf{B}\vec{x}$ is in the nullspace of $\mathbf{A}$ ($\mathbf{B}\vec{x} = \vec{y} = \vec{0}$), which means that $\mathbf{B}\vec{x} = \vec{0}$ since $\mathbf{A}$ is full rank. So, the only vectors that are in the nullspace of $\mathbf{AB}$ are the vectors in the nullspace of $\mathbf{B}$, so $Null(\mathbf{AB}) = Null(\mathbf{B})$.

(c) (3 points) Conceptually, if a state-transition matrix has a non-trivial nullspace (i.e. non-zero) is the information about previous states preserved? Circle your answer, then give 1-2 sentences of justification.

Yes No Sometimes

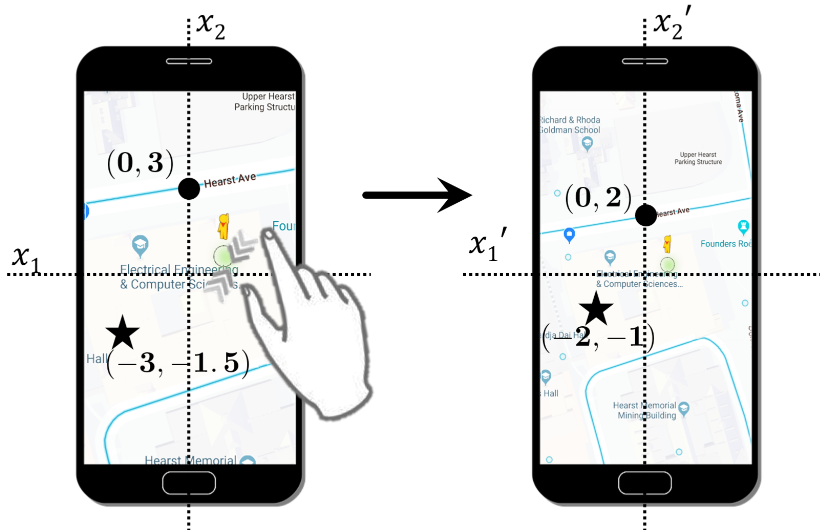
**Solution:** No. If the state-transition matrix has a non-trivial nullspace, then the matrix is non-invertible. If the matrix is non-invertible, then you cannot solve for a previous state using the state-transition matrix. Therefore, information about previous states is not fully preserved.

**8. Linear Algebra in a Pinch! (16 points)**

As you look at your smartphone to get Google Maps directions to TeaOne in Cory Hall, you realize that your smartphone is doing exactly what you do in class–linear algebra! As your fingers apply gestures to the screen, the phone is performing matrix transformations on the screen image.

In the following parts, we represent coordinates before transformation as $\vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ and coordinates after transformation as $\vec{x'} = \begin{bmatrix} x_1' & x_2' \end{bmatrix}^T$. Although specific points are labeled, the transformation applies to all points in the $(x_1, x_2)$ plane.

(a) (5 points) You pinch the screen to zoom out, as shown in the figure below:




The point represented by a dot moves from $(0,3)$ to $(0,2)$, and the point represented by a star moves from $(-3,-1.5)$ to $(-2,-1)$. What is the transformation matrix, $\mathbf{A}$, such that $\vec{x'} = \mathbf{A}\vec{x}$ for this transformation?

**Solution:** Looking at each dimension, we see that both $x_1'$ and $x_2'$ are scaled by $\frac{2}{3}$. Mathematically, the transformation can be written as a system of equations with 4 unknowns. Using the given two points, the matrices are:

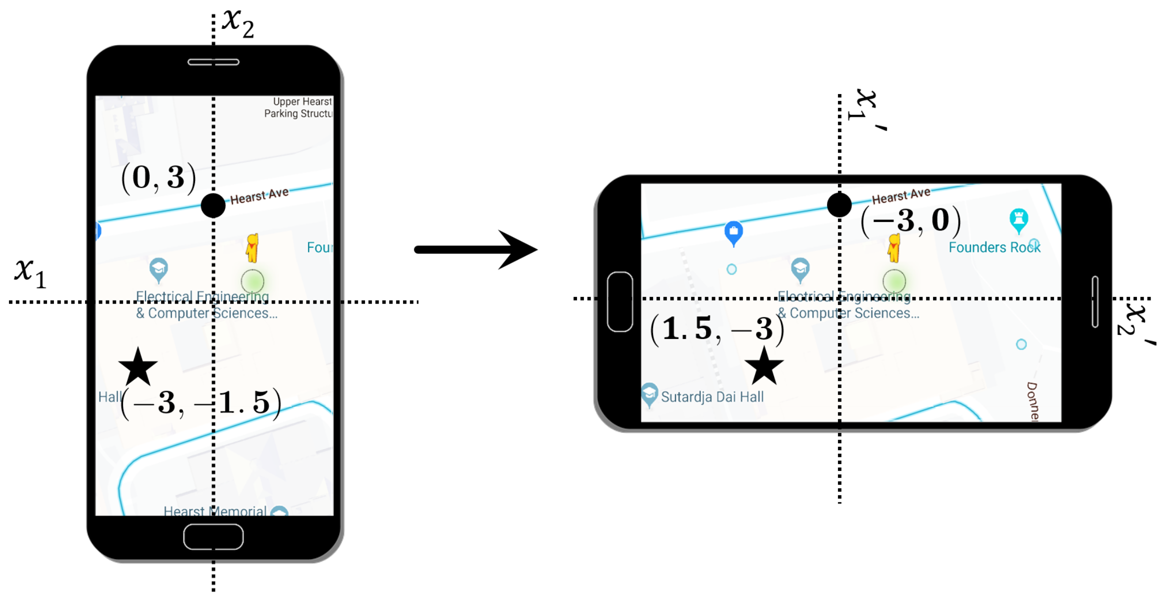
$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$$

These lead to the following equations: $3b = 0$, $3d = 2$, $-3a - 1.5b = -2$, and $-3c - 1.5d = -1$

Therefore $a = \frac{2}{3}, b = 0, c = 0$ and $d = \frac{2}{3}$, and the resulting tranformation matrix is $\mathbf{A} = \begin{bmatrix} 2/3 & 0 \\ 0 & 2/3 \end{bmatrix}$

(b) (5 points) Smartphones are smart, so if you rotate your phone, the map will reorient itself to make sure you can read it! That is, if you rotate your phone 90° clockwise, the map will rotate 90° counter-clockwise relative to the phone, as shown below.




The point represented by a dot moves from $(0,3)$ to $(-3,0)$, and the point represented by a star moves from $(-3,-1.5)$ to $(1.5,-3)$. What is the transformation matrix, $\mathbf{R}$, such that $\vec{x'} = \mathbf{R}\vec{x}$ for this transformation?

**Solution:** Our rotation matrix is of the form $\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, where $\theta$ is the angle the image as been rotated. Since we are rotating the phone 90° clockwise, this corresponds to $\theta = 90°$ on our coordinate system, giving us:

$$\mathbf{R} = \begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

A similar method used in the previous part can also be used to solve the problem: namely solving a system of four unknowns. Using the two points, the matrices are:

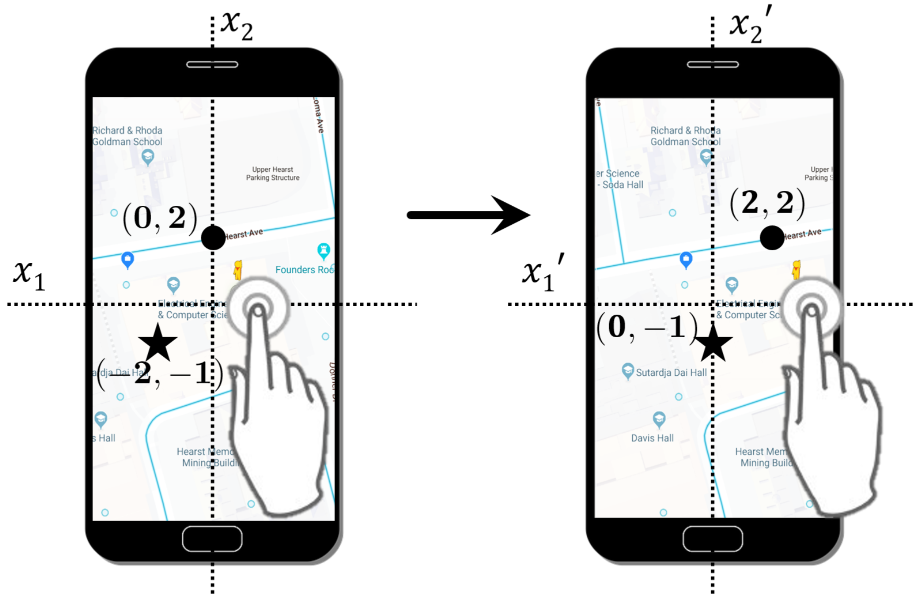
$$\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1.5 \\ -3 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -3 \\ -1.5 \end{bmatrix}$$

Leading to equations: $3b = -3$, $3d = 0$, $-3a - 1.5b = 1.5$, and $-3c - 1.5d = -3$, and

$$\mathbf{R} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(c) (6 points) So far, we have only done transformations that involve rotation and scaling, but another key feature of Google Maps is that we can scroll laterally across a map, as shown below.




The point represented by a dot moves from $(0,2)$ to $(2,2)$, and the point represented by a star moves from $(-2,-1)$ to $(0,-1)$.

In the previous parts we were able to represent the map transformations in the form

$$\vec{x}' = \mathbf{A}\vec{x} + \vec{b}$$

(Previously $\vec{b} = \vec{0}$.) What are $\mathbf{A}$ and $\vec{b}$ for the scrolling operation above? Is this a linear transformation?

**Solution:** The scrolling operation is represented by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

This is not a linear transformation since $\vec{b} \neq \vec{0}$. One way to see this is that an input of $(0,0)$ does not yield an output of $(0,0)$.

Fun fact! If you directly plug in the points' coordinates and solve for a single transformation matrix, $\mathbf{A}$, you can by chance find a solution that fits the two points explicitly given! However, the scrolling operation should be true for all points on the map, and without $\vec{b}$, this is not possible. Once again, you can show this by testing $(0,0)$, which does not work.

**9. Mining Population (20 points)**

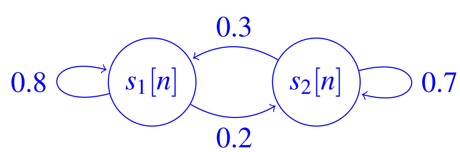
There is a population of cryptocurrency miners. These miners are primarily interested in two coins: OskiCoin and BearCoin, and they switch between mining the two coins in a predictable way. Every week, 20% of OskiCoin miners switch to BearCoin and 30% of BearCoin miners switch to OskiCoin. The remaining miners keep mining the same coin for the following week.

Let $s_1[n]$ be the number of miners of OskiCoin on week $n$ and $s_2[n]$ be the number of miners of BearCoin on week $n$.

$$\vec{s}[n] = \begin{bmatrix} s_1[n] \\ s_2[n] \end{bmatrix}$$

(a) (2 points) Draw a well-labeled directed graph showing how the population of miners changes each week. Be sure to label each node and place appropriate weights on each edge.

**Solution:**




(b) (2 points) Determine the state transition matrix **A**.

**Solution:** The state transition matrix describes the relationship of the state vector from one timestep to the next:

$$\vec{s}[n+1] = \mathbf{A}\vec{s}[n]$$

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

(c) (5 points) Find the eigenvalues ($\lambda_1 \ldots \lambda_n$) and eigenvectors ($\vec{v}_1 \ldots \vec{v}_n$) of **A**.

**Solution:**
We can write out the characteristic polynomial for A, or we can determine the eigenvalues by inspection. We denote the eigenvalues by $\lambda_1$ and $\lambda_2$, in decreasing order.

$$\begin{aligned} (0.8-\lambda)(0.7-\lambda) - 0.06 &= 0 \\ 0.56 - 0.7\lambda - 0.8\lambda + \lambda^2 - 0.06 &= 0 \\ 0.5 - 1.5\lambda + \lambda^2 &= 0 \end{aligned}$$

$$\lambda = \frac{1.5 \pm \sqrt{2.25 - 4(0.5)}}{2}$$
$$= \frac{1.5 \pm 0.5}{2}$$
$$\lambda_1 = 1$$
$$\lambda_2 = 0.5$$

Once we've determined the eigenvalues, we can solve for the eigenvectors. We start by solving for the eigenvector $\vec{v}_1$ corresponding to $\lambda_1 = 1$:

$$\mathbf{A} - \mathbf{I} = \begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$$
$$\implies \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

We proceed with solving for the eigenvector $\vec{v}_2$ corresponding to $\lambda_2 = 0.5$:

$$\mathbf{A} - 0.5\mathbf{I} = \begin{bmatrix} 0.3 & 0.3 \\ 0.2 & 0.2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$
$$\implies \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(d) (5 points) Express $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ in the coordinate system of the eigenvectors, $\mathbf{V} = \{\vec{v}_1, \ldots, \vec{v}_n\}$.

**Solution:**
We can solve this problem by inspection or by computation. By inspection:

$$\begin{bmatrix} 5 \\ 0 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2$$

where $\vec{v}_1$ and $\vec{v}_2$ are the eigenvectors determined in the solution to the previous part.
Alternatively, note that this is a change of basis problem. We are looking for weights $u_1$ and $u_2$ such that

$$\vec{v}_1 u_1 + \vec{v}_2 u_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

.
We can rewrite this equation as

$$\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

.

We can simply solve for $u_1$ and $u_2$ through Gaussian elimination.
We plug in the values for the eigenvectors and rewrite the equation as an augmented matrix:

$$\left[\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & -1 & 0 \end{array}\right] \underbrace{\Longrightarrow}_{R_2 \leftarrow R_1 + R_2} \left[\begin{array}{cc|c} 3 & 1 & 5 \\ 5 & 0 & 5 \end{array}\right]$$
$$\underbrace{\Longrightarrow}_{\text{swap } R_1, R_2} \left[\begin{array}{cc|c} 5 & 0 & 5 \\ 3 & 1 & 5 \end{array}\right]$$
$$\underbrace{\Longrightarrow}_{R_1 \leftarrow R_1/5} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 3 & 1 & 5 \end{array}\right]$$
$$\underbrace{\Longrightarrow}_{R_2 \leftarrow R_2 - 3R_3} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

In both cases, we can see that $u_1 = 1$, and $u_2 = 2$, for our eigenvectors $\vec{v}_1$ and $\vec{v}_2$.

Your answer may vary, depending on the magnitude of your eigenvalues. As long as $\vec{v}_1 u_1 + \vec{v}_2 u_2 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, then you will receive full credit.

(e) (6 points) If we start with 1000 miners mining OskiCoin and 0 miners mining BearCoin, then what will the steady state distribution of miners be?

**Solution:** Find $\lim_{n\to\infty} \vec{s}[n]$.

Note that we can rewrite $\vec{s}[0]$ as $200 \begin{bmatrix} 5 \\ 0 \end{bmatrix}$. This means we can reuse our result from the previous part to this problem.

$$\begin{aligned} \vec{s}[n] &= \mathbf{A}^n \vec{s}[0] \\ \vec{s}[n] &= \mathbf{A}^n \left( 200 \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right) \\ &= \mathbf{A}^n \cdot 200 \left( \vec{v}_1 + 2\vec{v}_2 \right) \\ &= 200 \cdot \mathbf{A}^n \vec{v}_1 + 400 \cdot \mathbf{A}^n \vec{v}_2 \\ &= 200 \cdot \lambda_1^n \vec{v}_1 + 400 \cdot \lambda_2^n \vec{v}_2 \end{aligned}$$

Note that $\lambda_1 = 1$ and $\lambda_2 = 0.5$. This tells us:

$$\lim_{n\to\infty} \lambda_1^n = 1$$
$$\lim_{n\to\infty} \lambda_2^n = 0$$

We can drop off the second term of the summation, leaving us with:

$$\lim_{n \to \infty} \vec{s}[n] = 200 \cdot \vec{v}_1$$
$$= \begin{bmatrix} 600 \\ 400 \end{bmatrix}$$

Therefore, in the steady state, there are 600 miners mining OskiCoin, and 400 miners mining BearCoin.

**10. Bad Barometers (20 points)**

Two scientists, Alice and Bob, want to determine the air pressure at the bottom and the top of Mt. Diablo. They know that pressure changes linearly with altitude. We represent the air pressure at the top and bottom of the mountain as a vector:

$$\vec{p} = \begin{bmatrix} p_{bottom} \\ p_{top} \end{bmatrix}$$

Scientist Alice first takes a pressure measurement at the bottom of the mountain. Then she starts hiking up. 10% of the way up she she takes a second measurement. She knows she can invert her system of equations, so she turns around. We can represent Alice's measurements as

$$\vec{m}_a = \mathbf{A}_a\vec{p} \quad \text{where} \quad \mathbf{A}_a = \begin{bmatrix} 1 & 0 \\ 0.9 & 0.1 \end{bmatrix}$$

Scientist Bob also takes a pressure measurement at the bottom of the mountain. He starts hiking and is enjoying the view so he hikes 50% of the way to the top where he stops and takes a second measurement. We can represent Bob's measurements as

$$\vec{m}_b = \mathbf{A}_b\vec{p} \quad \text{where} \quad \mathbf{A}_b = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

Unfortunately, both scientists have old barometers that don't work very well, so rather than measuring the true pressure, they measure the pressure plus some offset, $\vec{s}$. We define $\vec{q}$ to be the calculated pressure based on the inaccurate measurement:

$$\vec{m}_a + \vec{s}_a = \mathbf{A}_a\vec{q}_a \qquad \vec{m}_b + \vec{s}_b = \mathbf{A}_b\vec{q}_b$$

We are interested in the error, $\vec{e}$, which is the difference between the true pressure and the calculated pressure:

$$\vec{e}_a = \vec{q}_a - \vec{p} \qquad \vec{e}_b = \vec{q}_b - \vec{p}$$

(a) (4 points) Determine an expression for $\vec{e}_a$ as a function of $\mathbf{A}_a$ and $\vec{s}_a$.

**Solution:**

Find expressions for $\vec{p}$ and $\vec{q}_a$:

$$\begin{aligned} \vec{p} &= \mathbf{A}_a^{-1}\vec{m}_a \\ \vec{q}_a &= \mathbf{A}_a^{-1}(\vec{m}_a + \vec{s}_a) \end{aligned}$$

Plug into the expression for $\vec{e}_a$:

$$\begin{aligned} \vec{e}_a &= \vec{q}_a - \vec{p} \\ \vec{e}_a &= \mathbf{A}_a^{-1}(\vec{m}_a + \vec{s}_a) - \mathbf{A}_a^{-1}\vec{m}_a \\ \vec{e}_a &= \mathbf{A}_a^{-1}\vec{s}_a \end{aligned}$$

(b) (6 points) Suppose that $\vec{s}_a = \vec{s}_b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. This means that our scientists measure pressure that is 1 kPa too low in their first measurement and 1 kPa too high in their second measurement. (Both scientist have very similar old barometers, so they have the same offset for both measurements.)

Calculate Alice's error, $\vec{e}_a$, and Bob's error, $\vec{e}_b$.

**Solution:** We use the expression found in Part (a). For Alice's error:

$$\mathbf{A}_a^{-1} = \begin{bmatrix} 1 & 0 \\ -9 & 10 \end{bmatrix}$$
$$\vec{e}_a = \mathbf{A}_a^{-1}\vec{s}_a = \begin{bmatrix} 1 & 0 \\ -9 & 10 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\vec{e}_a = \begin{bmatrix} -1 \\ 19 \end{bmatrix}$$

For Bob's error:

$$\mathbf{A}_b^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$
$$\vec{e}_b = \mathbf{A}_b^{-1}\vec{s}_b = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\vec{e}_b = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Bob's measurement matrix yields lower error than Alice's measurement matrix.

(c) (6 points) Consider a general invertible, diagonalizable measurement matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_1 \dots \lambda_n$ and corresponding eigenvectors $\vec{v}_1 \dots \vec{v}_n$, and a general offset vector $\vec{s} \in \mathbb{R}^n$. Show that

$$\vec{e} = \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i} \vec{v}_i$$

where $\alpha_1 \dots \alpha_n$ are scalar values such that $\sum_{i=1}^{n} \alpha_i \vec{v}_i = \vec{s}$.

**Solution:**

In Part (a) we showed

$$\vec{e} = \mathbf{A}^{-1}\vec{s}$$

We can substitute in the express for $\vec{s}$ given in the problem:

$$\vec{e} = \mathbf{A}^{-1} \sum_{i=1}^{n} \alpha_i \vec{v}_i$$
$$\vec{e} = \sum_{i=1}^{n} \alpha_i \mathbf{A}^{-1} \vec{v}_i$$

By the definition of eigenvalues and eigenvectors, we know

$$\lambda_i \vec{v}_i = \mathbf{A}\vec{v}_i$$
$$\mathbf{A}^{-1}\lambda_i \vec{v}_i = \vec{v}_i$$
$$\mathbf{A}^{-1}\vec{v}_i = \frac{1}{\lambda_i}\vec{v}_i$$

We can substitute this into the equation for $\vec{e}$. You may have also already known that the eigenvalues of $\mathbf{A}^{-1}$ are $\frac{1}{\lambda_i}$ in which case the derivation above is not necessary.

$$\vec{e} = \sum_{i=1}^{n} \alpha_i \frac{1}{\lambda_i} \vec{v}_i$$

Alternative solution:

We can diagonalize $\mathbf{A}$ as follows:

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^{-1}$$

where $\mathbf{V}$ as a matrix with the eigenvectors as it's columns, and $\mathbf{D}$ is a diagonal matrix with the corresponding eigenvalues along the diagonal. Plugging this into the expression for $\vec{e}$ from Part (a) gives

$$\begin{aligned}
\vec{e} &= \mathbf{A}^{-1}\vec{s} \\
\vec{e} &= \left(\mathbf{V}\mathbf{D}\mathbf{V}^{-1}\right)^{-1}\vec{s} \\
\vec{e} &= \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^{-1}\vec{s}
\end{aligned}$$

We define $\vec{\alpha}$ as a vector that contains $\alpha_1 \dots \alpha_n$. The following expressions are equivalent:

$$\begin{aligned}
\sum_{i=1}^{n} \alpha_i \vec{v}_i &= \vec{s} \\
\mathbf{V}\vec{\alpha} &= \vec{s}
\end{aligned}$$

We substitute this expression for $\vec{s}$ into the expression for $\vec{e}$ above.

$$\begin{aligned}
\vec{e} &= \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^{-1}\vec{s} \\
\vec{e} &= \mathbf{V}\mathbf{D}^{-1}\mathbf{V}^{-1}\mathbf{V}\vec{\alpha} \\
\vec{e} &= \mathbf{V}\mathbf{D}^{-1}\vec{\alpha}
\end{aligned}$$

Recalling that the inverse of a diagonal matrix is the inverse of each element

$$\begin{aligned}
\vec{e} &= \mathbf{V}\begin{bmatrix} \frac{\alpha_i}{\lambda_i} \\ \vdots \\ \frac{\alpha_n}{\lambda_n} \end{bmatrix} \\
\vec{e} &= \sum_{i=1}^{n} \frac{\alpha_i}{\lambda_i} \vec{v}_i
\end{aligned}$$

**Common Mistakes:**

- Stating $A^{-1} = \frac{1}{\lambda}$, $A = \lambda$, or $A^{-1} = \sum \frac{1}{\lambda_i}$. This commonly was seen when trying to plug in $A^{-1}$ while outside the summation. For example, stating $\vec{e} = A^{-1}\vec{s} = A^{-1}\sum \alpha_i \vec{v}_i = \frac{1}{\lambda}\sum \alpha_i \vec{v}_i$ is incorrect.
- $A^{-1} = VD^{-1}V^{-1}$ not $V^{-1}D^{-1}V$
- Trying to write A as a summation, such as stating $A = \sum \vec{v}_i \lambda_i \vec{v}_i^{-1}$. (Note that vectors do not have inverses)
- Assuming that diagonal matrices are commutative with non-diagonal matrices
- Assuming that $\vec{s}$ is an eigenvector of $A$

- Trying to split the summation incorrectly. For example, stating $\sum \frac{\alpha_i \vec{v}_i}{\lambda_i} = \frac{\sum \alpha_i \vec{v}_i}{\sum \lambda_i} = \frac{\vec{s}}{\sum \lambda_i}$ is not correct since summing fractions is not the same as summing the numerator and denominator separately.

(d) (4 points) You calculate the eigenvalues and eigenvectors for $\mathbf{A}_a$ and $\mathbf{A}_b$ to be the following:

| $\mathbf{A}_a$ | $\mathbf{A}_b$ |
|---|---|
| $\lambda_1 = 1$ | $\lambda_1 = 1$ |
| $\lambda_2 = 0.1$ | $\lambda_2 = 0.5$ |
| $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ | $v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ |

Based on the equation in Part (c) and the eigenvalues and eigenvectors above, which matrix ($\mathbf{A}_a$ or $\mathbf{A}_b$) is more sensitive to inaccurate measurements? In other words, which will have a larger error $\vec{e}$? Does this agree with the trend you saw in Part (b)?

**Solution:** $\mathbf{A}_a$ has a smaller eigenvalue so will have larger error according to the equation in Part (c), since the eigenvalues are in the denominator. Note that both $\mathbf{A}_a$ and $\mathbf{A}_b$ have the same eigenvectors and will therefore have the same $\alpha_i$ for a given noise vector.

In Part (b), $\mathbf{A}_a$ had much larger error, so this matches what we saw in Part (b).