Complete all problems. You must show your work or justify your answer for all problems, unless it is stated otherwise. You may quote theorems and results from class or the homework without justification (name the theorem or state "we proved in class that ..."). If you need more space, use the blank pages at the back of the exam. If you want me to grade work done any of those pages, clearly indicate this next to the appropriate problem.

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Name SOLUTIONS

Scratchwork. This page will not be graded.

1. Answer each question. You do NOT have to justify your answers.

   (a) (*2 points*) Let $G = \langle a \rangle$ be a cyclic group of order 20.

      i. The number of proper nontrivial subgroups of G is $\underline{\ 4\ }$.

      $20: \cancel{1}, 2, 4, 5, 10, \cancel{20}$

      ii. The order of $\langle a^{12} \rangle$ is $\underline{\ 5\ }$.

      $\frac{20}{\text{lcm}(12,20)} = \frac{20}{4} = 5$

   (b) *(2 points)* In the group $\mathbb{Z}_3 \times \mathbb{Z}_5$, the inverse of $(\bar{2}, \bar{3})$ is $\underline{(\bar{1}, \bar{2})}$.

   (c) *(2 points)* Define the *index* of a subgroup H in a group G.

   The number of left (or right) cosets of H in G.

   (d) *(2 points)* If H is a subgroup of a finite group G, then Langrange's theorem states that:

   the order of H divides the order of G.

   (e) *(2 points)* Let $\phi\colon \mathbb{Z}_8 \to \mathbb{Z}_4$ be the homomorphism defined by $\phi(\bar{1}) = \bar{1}$. Then the first isomorphism theorem (also known as the fundamental theorem of homomorphisms) states that which two groups are isomorphic?

   $\underline{\mathbb{Z}_8/\langle\bar{4}\rangle} \simeq \underline{\mathbb{Z}_4}$

   $\text{Ker}\,\phi = \{\bar{0}, \bar{4}\} = \langle\bar{4}\rangle$

   $\text{Im}\,\phi = \mathbb{Z}_4$

2. (*2 points each*) Consider the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 7 & 6 & 2 & 5 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 6 & 7 & 2 & 4 & 3 \end{pmatrix}$$

in S_7.

(a) Find the order of σ.

$\sigma = (1\ 3\ 7)(2\ 4\ 6\ 5)$

$\operatorname{lcm}(3,4) = \boxed{12}$

(b) Write $\sigma\tau^{-1}$ as a product of disjoint cycles and as a product of transpositions.

$\tau^{-1} = (1\ 2\ 5)(3\ 7\ 4\ 6)$

$$\boxed{\begin{aligned} \sigma\tau^{-1} &= (1\,4\,5\,3)(6\ 7) \\ &= (1\ 3)(1\ 5)(1\,4)(6\,7) \end{aligned}}$$

(c) Is $\sigma\tau^{-1} \in A_7$?

Yes, b/c it can be written as the product of an even # of transpositions

(d) How many cosets does $H = \langle\sigma\rangle$ have in S_7?

$|S_7| = 7!$

$|S_7 : H| = |S_7| / |H| = \frac{7!}{12} = \boxed{420}$

3. (*4 points*) Find a subgroup of order 8 in D_{10}, the dihedral group of a regular 10–gon, or prove that one does not exist.

Since 8 does not divide $20 = |D_{10}|$, Lagrange's thm implies that no such subgp exists.

4. *(6 points)* Suppose G is a group. For $a, b \in G$, define $a \sim b$ if there exists some $x \in G$ such that $a = xbx^{-1}$. Prove that $\sim$ is an equivalence relation on G.

- reflexive: Let $a \in G$. Then $a = eae^{-1}$ & $e \in G$, so $a \sim a$.
- symmetric: Let $a, b \in G$ s.t. $a \sim b$. Then $\exists x \in G$ s.t.

$$a = xbx^{-1} \Rightarrow b = x^{-1}ax$$
$$= x^{-1}a(x^{-1})^{-1}, \text{ \& } x^{-1} \in G, \text{ so } b \sim a.$$

- transitive: Let $a, b, c \in G$ s.t. $a \sim b$ & $b \sim c$. Then $\exists x, y \in G$ s.t. $a = xbx^{-1}$ & $b = ycy^{-1}$. Thus

$$a = x(ycy^{-1})x^{-1}$$
$$= (xy)\, c\, (y^{-1}x^{-1})$$
$$= (xy)\, c\, (xy)^{-1}, \text{ \& } xy \in G, \text{ so } a \sim c.$$

Therefore, $\sim$ is an equivalence relation.

5. *(6 points)* Let G be a group, and fix $g \in G$. Consider the map $\iota_g \colon G \to G$ defined by $\iota_g(x) = gxg^{-1}$. Prove that ι_g is an isomorphism.

- i_g is a homomorphism: let $x, y \in G$. Then
  $i_g(xy) = gxyg^{-1} = gxg^{-1}gyg^{-1} = i_g(x) i_g(y)$.
- injective: let $x, y \in G$ s.t. $i_g(x) = i_g(y)$. Then $gxg^{-1} = gyg^{-1}$, & by left & right cancellation, $x = y$.
- surjective: let $y \in G$. Then $g^{-1}yg \in G$, &
  $i_g(g^{-1}yg) = gg^{-1}ygg^{-1} = y$.

Therefore, i_g is an isomorphism.

6. *(6 points)* Let $\phi \colon G \to G'$ be a homomorphism between finite groups. Prove that $|\phi(G)|$ divides $|G|$ and $|G'|$.

- $|\varphi(G)|$ divides $|G'|$: We know that $\varphi(G) \leq G'$, so by Lagrange's Thm, $|\varphi(G)| \mid |G'|$.
- $|\varphi(G)|$ divides $|G|$: By the 1st Isomorphism thm, $G/\ker\varphi \simeq \varphi(G)$. Thus
$$|G/\ker\varphi| = |\varphi(G)|$$
$$|G|/|\ker\varphi| = |\varphi(G)|$$
$$|G| = |\ker\varphi||\varphi(G)|.$$

Since $|\ker\varphi| \in \mathbb{Z}^+$, it follows that $|\varphi(G)|$ divides $|G|$.

7. Let $G = \mathbb{Z}_{10} \times \mathbb{Z}_8$, and let $H = \langle(\bar{5}, \bar{4})\rangle$. For all parts of the problem, show any computations you do, but explanations in sentences are not required.

(a) (*2 points*) How many elements are in the quotient group G/H?

$|H| = \operatorname{lcm}(2,2) = 2$

$|G/H| = |G|/|H| = \frac{80}{2} = \boxed{40}$

(b) (*3 points*) Use the fundamental theorem of finitely generated abelian groups to list all possible isomorphism types for abelian groups of order $|G/H|$.

$40 = 2^3 \cdot 5$

- $\mathbb{Z}_8 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$
- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$

(c) (*3 points*) What is the order of $(\bar{5}, \bar{1}) + H$ in G/H?

$1 \cdot (\bar{5},\bar{1}) = (\bar{5},\bar{1}) \notin H$

$2 \cdot (\bar{5},\bar{1}) = (\bar{0},\bar{2}) \notin H$

$3 \cdot (\bar{5},\bar{1}) = (\bar{5},\bar{3}) \notin H$

$\vdots$

$7 \cdot (\bar{5},\bar{1}) = (\bar{5},\bar{7}) \notin H$

$8 \cdot (\bar{5},\bar{1}) = (\bar{0},\bar{0}) \in H \Rightarrow \boxed{|(\bar{5},\bar{1}) + H| = 8}$

(d) (*2 points*) Based on your answer to (c) and no additional calculations, either determine which group in (b) is isomorphic to G/H or explain why it is not possible to do so.

$\mathbb{Z}_8 \times \mathbb{Z}_5$ is the only gp in (b) which contains an elt of order 8, so $\boxed{G/H \cong \mathbb{Z}_8 \times \mathbb{Z}_5}$

8. *(5 points each)* Let $\phi\colon G \to G'$ be a homomorphism, and let $K = \ker\phi$.

(a) Prove that K is a subgroup of G.

- closed: Let $x, y \in \ker \phi$. Then $\phi(x) = \phi(y) = e'$, & so $\phi(xy) = \phi(x)\phi(y) = e'e' = e'$, so $xy \in \ker\phi$
- identity: Since ϕ is a hom, $\phi(e) = e' \Rightarrow e \in \ker\phi$.
- inverses: Let $x \in \ker\phi$. Then $\phi(x) = e'$, & so $\phi(x^{-1}) = \phi(x)^{-1} = (e')^{-1} = e' \Rightarrow x^{-1} \in \ker\phi$.

Therefore, $K \leq G$.

(b) Prove that K is a normal subgroup of G.

It suffices to show that $\forall g \in G$ & $\forall x \in K$, $gxg^{-1} \in K$.

Let $g \in G$ & $x \in K$. Then

$$\begin{aligned}\phi(gxg^{-1}) &= \phi(g)\phi(x)\phi(g)^{-1}\\ &= \phi(g)\, e'\, \phi(g)^{-1}\\ &= \phi(g)\phi(g)^{-1}\\ &= e'\end{aligned}$$

So $gxg^{-1} \in K$. Since g & x were arbitrary, this holds for all $g \in G$ & $\forall x \in K$. Therefore, $K \trianglelefteq G$.

9. (*2 points each*) Determine whether each of the following statements is true or false. You do not need to show any work and you do not need to justify your answers. Clearly write the word "True" or "False" for each part.

(a) A group of order 17 has no nontrivial proper subgroups.

true

(b) $\mathbb{Z} \simeq \mathbb{Q}$.

false

(c) In S_n, every cycle of odd length is an even permutation.

true

(d) Apart from the identity, all elements of $\mathbb{Z} \times \mathbb{Z}_{15}$ have infinite order.

false

(e) $\mathbb{Z}_{75} \simeq \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$

false

(f) If H and K are subgroups of a group G, then $H \cup K$ is a subgroup of G.

false

(g) The cosets of $\langle \bar{4} \rangle$ in $\mathbb{Z}_{12}$ are $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$, $\{\bar{4}, \bar{5}, \bar{6}, \bar{7}\}$, $\{\bar{8}, \bar{9}, \bar{10}, \bar{11}\}$.

false

(h) Every subgroup of a non-cyclic group is non-cyclic.

false

(i) $D_4 \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$.

false

(j) Let G be a group and $g \in G$ a fixed element. Then the map $\phi_g \colon G \to G$ defined by $\phi_g(x) = gx$ is a permutation of G.

true .

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