1.a) $A_1 = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}$, $e^{A_1 t} = \begin{bmatrix} e^t \cos 3t & e^t \sin 3t \\ -e^t \sin 3t & e^t \cos 3t \end{bmatrix}$ b) $A_2 = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}$ $e^{A_2 t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{4t} \end{bmatrix}$

c) $A_3 = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$, $e^{A_3 t} = \begin{bmatrix} e^{-3t} & te^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$ d) $A_4 = \begin{bmatrix} -2+j5 & 0 \\ 0 & -2-j5 \end{bmatrix}$, $e^{A_4 t} = \begin{bmatrix} e^{(-2+j5)t} & 0 \\ 0 & e^{(-2-j5)t} \end{bmatrix}$

e) $A_5 = \begin{bmatrix} 0 & -4 \\ 4 & 0 \end{bmatrix}$, $e^{A_5 t} = \begin{bmatrix} \cos 4t & -\sin 4t \\ \sin 4t & \cos 4t \end{bmatrix}$

2) poly with roots at $-2 \pm j1 \Rightarrow q(s) = (s+2-j1)(s+2+j1) = \cdots = s^2+4s+5$

Match with $p(s)$; $p(s) = s^2 + (-1+7\beta_1+5\beta_2)s + (1+4\beta_1+3\beta_2) = s^2+4s+5$

$\Leftrightarrow \begin{pmatrix} 7 & 5 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \Leftrightarrow \begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ $\begin{bmatrix} 7 & 5 \\ 4 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -5 \\ 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 3 & -5 \\ 4 & 7 \end{bmatrix}$

Hence $\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$ ///.

3) $\dot{x} = Ax + Ed + Bu$

$= Ax + Ed + B(Jz + Kr + Ly_m)$

$= Ax + Ed + B(Jz + Kr + L(Cx+n))$

$= \underline{(A+BLC)x + BJz + BKr + Ed + BLn}$

$\dot{z} = Fz + Gr + Hy_m$

$= Fz + Gr + H(Cx+n)$

$= \underline{HCx + Fz + Gr + 0 \cdot d + Hn}$

$\underline{y = Cx}$, $u = Jz + Kr + Ly_m$

$= Jz + Kr + L(Cx+n)$

$= \underline{LCx + Jz + Kr + 0 \cdot d + Ln}$

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ y \\ u \end{bmatrix} = \left[\begin{array}{c|c|c|c|c} A+BLC & BJ & BK & E & BL \\ \hline HC & F & G & 0 & H \\ \hline C & 0 & 0 & 0 & 0 \\ \hline LC & J & K & 0 & L \end{array}\right] \begin{bmatrix} x \\ z \\ r \\ d \\ n \end{bmatrix}$$

4) Let v_1, v_2, v_3 denote columns of V ; $\lambda_1, \lambda_2, \lambda_3$ denote diagonal entries of Λ

Note $\bar{v}_1 = v_2$ $\bar{\lambda}_1 = \lambda_2$. Write out $Av_1 = \lambda_1 v_1$

$A(v_1)_R = (\lambda_1)_R (v_1)_R - (\lambda_1)_I (v_1)_I$

$A(v_1)_I = (\lambda_1)_I (v_1)_R + (\lambda_1)_R (v_1)_I$

$A\left[(v_1)_R \ (v_1)_I\right] = \left[(v_1)_R \ (v_1)_I\right]\begin{bmatrix} (\lambda_1)_R & (\lambda_1)_I \\ (-\lambda_1)_I & (\lambda_1)_R \end{bmatrix}$

We derived this in class, in general, and also had quiz and homework on it. Since $\bar{v}_2 = v_1$, $\bar{\lambda}_2 = \lambda_1$, the equation $Av_2 = \lambda_2 v_2$ gives the same equations

Hence

$$A \underbrace{\begin{bmatrix} (v_1)_R & (v_1)_I & v_3 \end{bmatrix}}_{W} = \begin{bmatrix} (v_1)_R & (v_1)_I & v_3 \end{bmatrix} \underbrace{\begin{bmatrix} (\lambda_1)_R & (\lambda_1)_I & 0 \\ -(\lambda_1)_I & (\lambda_1)_R & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}}_{\Gamma}$$

$$W = \begin{bmatrix} -1 & 4 & 0 \\ 2 & -1 & 1 \\ 6 & 0 & -1 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} -2 & 2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

5) a) closed loop is

$$\dot{x} = (A - B_2 K_p C)x + B_1 d + B_2 K_p r$$
$$y = Cx$$

Since $B_2 C \neq 0$, you can choose K_p to set $A - BK_pC$ <u>arbitrarily</u>. Hence Goal 1 & 2 achievable. $SSG_{r \to y} = \dfrac{B_2 K_p C}{B_2 K_p C - A}$. This equals 1 if and only if $A = 0$. Also $SSG_{d \to y} = \dfrac{B_1 C}{B_2 K_p C - A} \neq 0$. Hence Goal 4 (and hence 6) <u>not</u> achievable. If $A = 0$, then (3,5) achievable. If $A \neq 0$, then (3,5) <u>not</u> achievable.

b) closed loop $\dot{x} = (A + B_2 K_2 C)x + B_1 d + B_2 K_1 r$. <u>Goal 1 & 2 achievable with K_2</u>

$$y = Cx$$

$$SSG_{r \to y} = \frac{B_2 K_1 C}{B_2 K_2 C - A}$$ so <u>goal #3 achievable</u>. (5) is <u>not</u> achievable

4 & 6 are <u>not</u>, similar to part (a)

c) closed-loop is

$$\begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} = \begin{bmatrix} A & B_2 K_I \\ -C & 0 \end{bmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} 0 & B_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix}$$

for closed loop, char poly is $\lambda(\lambda - A) + B_2 K_I C = \lambda^2 - A\lambda + B_2 K_I C$

Clearly cannot do Goal 2, since 2 evals, only 1 parameter (K_I). If $A<0$, then $B_2 K_I C > 0$ results in stability. If $A \geq 0$, then always unstable.

If closed-loop is stable, then $SSG_{r \to y} = 1$, $SSG_{d \to y} = 0$ (check matrices, due to integral action) and this is robust (#3, 4, 5, 6). If closed-loop is unstable, then none. So

$$A < 0 \Rightarrow \text{Goals } (1, 3, 4, 5, 6) \text{ achievable}$$
$$A \geq 0 \Rightarrow \text{none achievable}$$

d) This is exactly the problem considered in section 13.1. So all goals achievable.

6) Closed loop is

$$\begin{pmatrix} \dot{x} \\ \dot{q} \end{pmatrix} = \begin{bmatrix} -4-6K_p & 3K_I \\ -2 & 0 \end{bmatrix} \begin{pmatrix} x \\ q \end{pmatrix} + \begin{pmatrix} 3K_p & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ d \end{pmatrix}$$

$$y = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{pmatrix} x \\ q \end{pmatrix}$$

Char eq $= (\lambda + 4 + 6K_p)\lambda + 6K_I$

$$= \lambda^2 + \underbrace{(4 + 6K_p)}_{2\zeta\omega_n}\lambda + \underbrace{6K_I}_{\omega_n^2} \Longrightarrow K_p = \frac{2\zeta\omega_n - 4}{6}$$

$$\boxed{K_p = \frac{\zeta\omega_n - 2}{3}}$$

$$\boxed{K_I = \frac{\omega_n^2}{6}}$$

$\zeta = .9$, $\omega_n = 10$ $\Rightarrow$ $\boxed{K_p = 7/3 \quad K_I = \frac{100}{6} = \frac{50}{3}}$

b) 1 (section 13.1)

c) 0 (section 13.1)

d) $-CA^{-1}B$ where $A = \begin{bmatrix} -18 & 50 \\ -2 & 0 \end{bmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$C = [-14/3 \quad 50/3]$

$$A^{-1}B = \begin{bmatrix} 0 \\ 2/100 \end{bmatrix}$$

$$\boxed{\Rightarrow SSG_{d \to y} = -1/3}$$