NAME:

Physics 112
Spring 2004

Midterm 2. Solution

Monday 29 March 2004, 8:10-9:00 a.m.

50 points=50minutes

1) Equilibrium with exchange of particles (10 points)

We consider electronic impurities in a semiconductor at temperature τ. Depending on their nature, each of such electronic impurities can either donate or accept electrons. There are in equilibrium with the large bath of free electrons in the crystal.

a) (4 points) Let us first consider a mono-valent donor site (it can only donate one electron). If it is ionized (no electron), its energy is ε_+. There are two states of energy ε_0 where the site is neutral (one with the electron spin up and the other one with spin down). Write down the grand partition and the fraction of ionized sites in terms of the temperature and the chemical potential.

$$\mathfrak{z} = e^{-\varepsilon_+/\tau} + 2\cdot e^{-(\varepsilon_0-\mu)/\tau}$$

$$f_+ = \frac{1}{\mathfrak{z}}\cdot e^{-\varepsilon_+/\tau}$$

b) (3 points) Let us then consider a mono-valent acceptor site(it can only accept one electron). If it is not ionized (no electron), its energy is ε_0. There are two states of energy ε_- where the site is ionized (one with the electron spin up and the other one with spin down). Write down the grand partition and the fraction of ionized sites.

$$\mathfrak{z} = e^{-\varepsilon_0/\tau} + 2\cdot e^{-(\varepsilon_--\mu)/\tau}$$

$$f_- = \frac{1}{\mathfrak{z}}\cdot 2e^{-(\varepsilon_--\mu)/\tau}$$

c) (3 points) How is the chemical potential determined?

By the concentration of electrons in the lattice.

2) **Thermodynamic identity** (10 points)

We will consider a system of N particles (we assume for simplicity that there is only one species of particles) and study the Gibbs free energy defined with our usual notations as

$$G = U - \tau\sigma + PV$$

a) (4 points) Write down the thermodynamic differential identity for G

$$dG = dU - \tau d\sigma - \sigma d\tau + p dV + V dp$$

$$= -\sigma d\tau + V dp + \mu dN \quad , \text{ using } dU = \tau d\sigma - p dV + \mu dN .$$

What are the natural variables for G? For what kind of systems can G be defined?

$\therefore G = G(\tau, p, N)$

This is well defined for constant pressure in thermal equilibrium.

b) (3 points) Show that the chemical potential expressed in terms of the temperature, the pressure and number of particles do not depend on the number of particles in the system. (Hint: divide a system at equilibrium into two unequal systems)

$$\mu(\tau, P, N) = \mu(\tau, P)$$

τ_1 P_1 $N1$ $\mu_1(\tau_1, P_1, N_1)$	τ_2 P_2 N_2 $\mu_2(\tau_2, P_2, N_2)$

From the definition $\mu = \left.\frac{\partial G(\tau, p, N)}{\partial N}\right|_{\tau, p}$, it is eay to see that $\mu = \mu(\tau, p, N)$. Let's imagine that we arbitrarily divide our system into two parts, then $\tau_1 = \tau_2$, $p_1 = p_2$ and $\mu_1 = \mu_2$. Otherwise it will not be in equilibrium. Because $\mu = \mu(\tau, p, N)$, $\mu_1(\tau_1, p_1, N_1) = \mu_2(\tau_2, p_2, N_2)$ and $N_1 \neq N_2$, μ should be independent of N.

$$\therefore \mu(\tau, p, N) = \mu(\tau, p)$$

c) (3 points) Using the fact that G=0 when no particle is present, show that

$$G = \mu(\tau, P) N$$

$$\text{or } U - \tau\sigma + PV + \mu N \equiv 0$$

(Hint: integrate the thermodynamic identity)

By integrating the definition $\left.\frac{\partial G}{\partial N}\right|_{\tau, p} = \mu(\tau, p)$, we get

$G = \mu(\tau, p) \cdot N + c$. By imposing B.C. $N = 0 \rightarrow G = 0$

3) **Pressure of a photon gas** (15 points)

We consider a photon gas in equilibrium at temperature τ in a volume V.

a) (5points) Using our partition function method, show that the Helmoltz free energy of a mode of angular frequency ω is

$$F_\omega = \tau \log\left(1-\exp\left(-\frac{\hbar\omega}{\tau}\right)\right)$$

$$Z = 1 + e^{-\hbar\omega/\tau} + e^{-2\hbar\omega/\tau} + \cdots = \frac{1}{1-e^{-\hbar\omega/\tau}}$$

$$\therefore\ F_\omega = -\tau \log Z = \underline{\tau \log(1-e^{-\hbar\omega/\tau})}$$

b) (7 points)From what we know of the density of (spatial) states in phase space and the number of photon polarization, show that the free energy of the full set of modes is

$$F = -V\int \frac{\hbar\omega^3 d\omega}{3\pi^2c^3\left(\exp\left(\frac{\hbar\omega}{\tau}\right)-1\right)}$$

(Hint: integrate by part!). How is this related to the energy density?

$$\left\{\begin{array}{l}\int d^3x = V \\ d^3p = p^2 dp\, d\Omega_p \\ p = \frac{\hbar\omega}{c},\ \omega^2 d\omega = \frac{1}{3} d(\omega^3)\end{array}\right.$$

$$F = \frac{2}{h^3}\int d^3x\, d^3p \cdot \tau \log\left(1-e^{-\hbar\omega/\tau}\right)$$

$$= \frac{2V\cdot 4\pi\tau}{h^3}\left(\frac{\hbar}{c}\right)^3 \int \omega^2 d\omega \log\left(1-e^{-\hbar\omega/\tau}\right)$$

$$= \frac{8\pi V\tau}{h^3}\left(\frac{\hbar}{c}\right)^3 \frac{\omega^3}{3}\log\left(1-e^{-\hbar\omega/\tau}\right)\Big|_0^\infty - \frac{8\pi V\tau}{h^3}\left(\frac{\hbar}{c}\right)^3\left(\frac{\hbar}{\tau}\right)\int \frac{\omega^3 d\omega}{3}\cdot\frac{1}{e^{\hbar\omega/\tau}-1}$$

$$= \underline{-V\int \frac{\hbar\omega^3 d\omega}{3\pi^2c^3\cdot(e^{\hbar\omega/\tau}-1)}}$$

From the expression of photon energy density

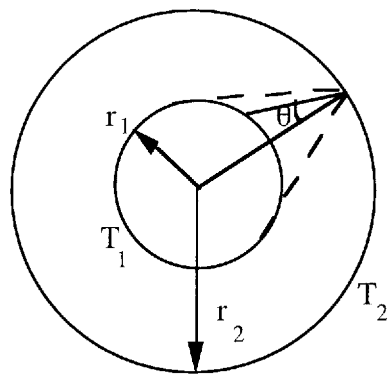
$$u = \frac{U}{V} = \frac{\hbar}{\pi^2c^3}\int \frac{\omega^3 d\omega}{e^{\hbar\omega/\tau}-1} \Rightarrow \underline{\underline{F = -\frac{1}{3}uV}} = -\frac{1}{3}U$$

c) (3 points) From this result, compute the pressure and relate it to the energy density.

$$p = -\frac{\partial F}{\partial V}\Big|_\tau = \int \frac{\hbar\omega^3 d\omega}{3\pi^2c^3(e^{\hbar\omega/\tau}-1)} = \frac{1}{3}\left(\frac{U}{V}\right) = \frac{1}{3}u$$

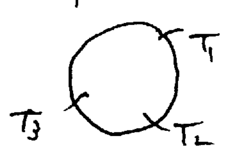
4) **Detailed Balance and Black Body Radiation** (15 points)

Consider two concentric spherical shells that are black body radiators. One has a radius r_1 and fixed temperature T_1 and the other has radius r_2 and fixed temperature T_2.



a) (5 points) Consider first the case where there is no shell inside. Describe how, at equilibrium, the energy emitted towards the inside by each point of the sphere is related to the energy absorbed at the same point. Show that even if the sphere is not thermally conducting, it will tend to equilibrate at a given temperature.

At equilibrium, the energy emitted towards inside by one point is same as the energy absorbed at that point from other points of the shell.



If there are several points which have different temperature, then the low temperature point will receive more energy than it will emit. So it will achieve same temperature.

b) (6 points) If $T_1 = T_2$, how is the power absorbed by the inner sphere related to the power it emits? Deduce from this result what is the fraction f of the radiation emitted by shell 2 which is absorbed by shell 1 (without calculating the integral over the angle). Check your result by taking the limit r_1=0.

Net rate of heat flow (net power) $= \sigma_B T_2^4 \cdot 4\pi r_2^2 - \sigma_B T_1^4 \cdot 4\pi r_1^2$

$= \sigma_B T^4 \cdot 4\pi (r_2^2 - r_1^2)$

$$f = 1 - \text{fraction of net power} = 1 - \frac{\sigma_B T^4 \cdot 4\pi (r_2^2 - r_1^2)}{\sigma_B T^4 \cdot 4\pi r_2^2} = \left(\frac{r_1}{r_2}\right)^2$$

c) (4 points) Derive from this result the expression of the power $P_{2\to 1}$ emitted by the outer shell 2 and absorbed by shell 1 when the outer shell is at temperature $T_2 \neq T_1$. What is the net heat flow between the two shells if $T_1 \neq T_2$?

From part b) we can see that the power absorbed by the inner shell is the fraction $\left(\frac{r_1}{r_2}\right)^2$ of the power emitted by the outer shell.

$$\therefore P_{2\to 1} = \left(\frac{r_1}{r_2}\right)^2 \cdot (4\pi r_2^2)\, \sigma_B T_2^4 = \underline{\underline{4\pi r_1^2 \cdot \sigma_B T_2^4}}.$$

$\therefore$ the net heat flow

$$= 4\pi r_1^2 \cdot \sigma_B T_2^4 - 4\pi r_1^2 \cdot \sigma_B T_1^4$$

$$= \underline{\underline{4\pi r_1^2 \cdot \sigma_B \left(T_2^4 - T_1^4\right)}}$$