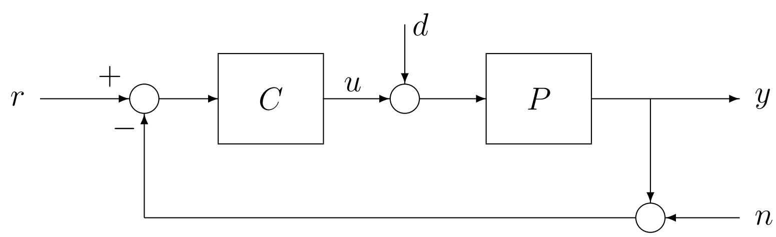
## 30.3 Fall 2017 Final

| # 1 | # 2 | # 3 | # 4 | # 5 | # 6 | # 7 | # 8 | NAME |
|---|---|---|---|---|---|---|---|---|
| 20 | 15 | 20 | 15 | 15 | 18 | 15 | 20 | |

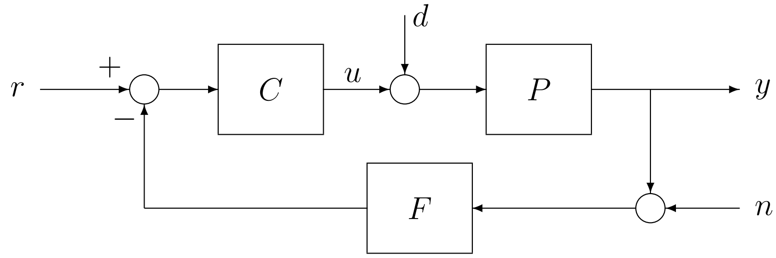
| # 9 | # 10 | # 11 | # 12 | # 13 | # 14 | # 15 | # 16 |
|---|---|---|---|---|---|---|---|
| 18 | 12 | 12 | 15 | 12 | 20 | 18 | 15 |

**Facts:**

1. **3rd order stability test:** All roots of the third-order polynomial $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$ have negative real-parts if and only if $a_1 > 0, a_3 > 0$ and $a_1 a_2 > a_3$.
2. **4ht order stability test:** All roots of the fourth-order polynomial $\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4$ have negative real-parts if and only if $b_1 > 0, b_4 > 0$, $b_1 b_2 > b_3$ and $(b_1 b_2 - b_3) b_3 > b_1^2 b_4$.
3. If $u(t) = \bar{u}$ (a constant) for all $t \geq 0$, and $A \in \mathbb{R}^{n \times n}$ is invertible, then the response of $\dot{x}(t) = Ax(t) + Bu(t)$ from initial condition $x(0) = x_0$ is $x(t) = e^{At}x_0 + (e^{At} - I)A^{-1}B\bar{u}$.
4. The characteristic polynomial of the 1st order (vector) differential equation $\dot{x}(t) = Ax(t)$ is $\det(\lambda I_n - A)$, where $n$ is the dimension of $x$.
5. The block diagram below is referred to as the "standard $(P, C)$ feedback loop",




6. The block diagram below is referred to as the "standard $(P, C, F)$ feedback loop",



1. Consider the following plant/controller transfer function pairs, for the standard $(P, C)$ feedback configuration (see front page)

$$P_1 = \frac{1}{s^2}, \quad C_1 = \frac{s+\frac{1}{3}}{s+3}, \qquad P_2 = \frac{1}{s(s+10)}, \quad C_2 = \frac{20s+8}{s}$$

$$P_3 = \frac{1}{(s-1)(s+20)}, \quad C_3 = \frac{16(4s+1)}{s}, \qquad P_4 = \frac{1}{s^2}, \quad C_4 = \frac{K_P s + K_I}{s}$$

   (a) What is the closed-loop characteristic polynomial for the $(P_1, C_1)$ pair

   (b) Is the $(P_1, C_1)$ closed-loop system stable?

   (c) What is the closed-loop characteristic polynomial for the $(P_2, C_2)$ pair

   (d) Is the $(P_2, C_2)$ closed-loop system stable?

   (e) What is the closed-loop characteristic polynomial for the $(P_3, C_3)$ pair

   (f) Is the $(P_3, C_3)$ closed-loop system stable?

   (g) What is the closed-loop characteristic polynomial for the $(P_4, C_4)$ pair

   (h) Are there any values of $K_P, K_I$ such that closed-loop $(P_4, C_4)$ system is stable?

   (i) True/False: All 2nd order plants can be stabilized by a PI controller

   (j) True/False: Some unstable 2nd order plants can be stabilized by a PI controller

2. A stable linear system has a frequency-response function, denoted $H(\omega)$.

   (a) It is known that $H(2) = 3 - 1j$. What precise/concrete statement can be made about a particular response of the system?

   (b) The state-space model for the system is the usual

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

   where $u$ and $y$ are the input, and output, respectively. How are $A, B, C, D$ and $H(\omega)$ related?

   (c) The transfer function of the system is denoted $G(s) = \frac{n(s)}{d(s)}$. How is the transfer function related to the frequency-response function?

3. The equations for a satellite orbiting a stationary mass are

$$\dot{x}_1(t) = x_2(t), \qquad \dot{x}_2(t) = x_1(t)x_3^2(t) - \frac{\beta}{x_1^2(t)}, \qquad \dot{x}_3(t) = \frac{1}{x_1(t)}\left(u(t) - 2x_2(t)x_3(t)\right),$$

where $x_1$ is the radius, $x_2$ is the rate-of-change of radius, and $x_3$ is the orbital angular velocity. The control $u$ is a force applied in the tangential direction. The gravitational constant and mass of the stationary mass are collected into the single positive constant $\beta$.

   (a) Show that for any constant $\bar{x}_1 > 0$, there exist constants $\bar{x}_2, \bar{x}_3$ and $\bar{u}$ such that

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}, \quad \bar{u}$$

   is an equilibrium point of the system. (Note - you can choose the equilibrium point that has $\bar{x}_3 > 0$).

   (b) Find the Jacobian linearization of the system about the equilibrium point.

4. Suppose a linear system $\dot{x}(t) = Ax(t) + Bu(t)$ has state-space data

$$A = \begin{bmatrix} 0 & 1 & 0 \\ a & 0 & b \\ 0 & -c & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ d \end{bmatrix},$$

where $a, b, c, d$ are <u>positive</u> constants.

   (a) What is the characteristic polynomial of $A$?

   (b) Given only the information that $a, b, c, d$ are all positive, can you conclude anything about the stability (ie., all eigenvalues have negative real-parts) of the system?

   (c) Define an output $y(t) = Cx(t)$ with

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

   What is the transfer function from $u$ to $y$?

5. Take the plant $P$, described by its transfer function,

$$P(s) = \frac{s^2 - 4}{s(s^2 + 1)}$$

which is quite challenging to control.

(a) Let $u$ and $y$ denote the input and output, respectively. What is the differential equation governing the relationship between $u$ and $y$

(b) Consider the standard $(P, C)$ closed-loop configuration (see front page for definition). Using a proportional controller, $C(s) = K_P$, a constant-gain, can the closed-loop system be made stable, by proper choice of $K_P$?

(c) In terms of transfer functions, what is the simplest controller form, $C(s)$ you can propose, such that <u>if</u> the controller coefficients are chosen properly, will render the closed-loop system (using standard $(P, C)$ closed-loop configuration) stable?

6. Consider the standard $(P, C)$ configuration. Let the transfer functions of $P$ and $C$ be denoted

$$P(s) = \frac{n_P(s)}{d_P(s)}, \qquad C(s) = \frac{n_C(s)}{d_C(s)}$$

Match the defined transfer functions $M_i$ or $N_i$ with the closed-loop transfer functions (eg., $G_{r\to y}$ means closed-loop transfer function from $r$ to $y$)

$$M_1 := \frac{n_P n_C}{d_P d_C + n_P n_C}, \quad M_2 := \frac{d_P d_C}{d_P d_C + n_P n_C}, \quad M_3 := \frac{n_P d_C}{d_P d_C + n_P n_C}, \quad M_4 := \frac{d_P n_C}{d_P d_C + n_P n_C}$$

$$N_1 = -M_1, \quad N_2 = -M_2 \quad N_3 = -M_3 \quad N_4 = -M_4$$

(a) $G_{r\to y} =$ ________

(b) $G_{r\to u} =$ ________

(c) $G_{d\to y} =$ ________

(d) $G_{d\to u} =$ ________

(e) $G_{n\to y} =$ ________

(f) $G_{n\to u} =$ ________

7. Consider the standard $(P, C, F)$ configuration. Let the transfer functions of $P$ and $C$ be denoted
$$P(s) = \frac{n_P(s)}{d_P(s)}, \qquad C(s) = \frac{n_C(s)}{d_C(s)}, \qquad F(s) = \frac{n_F(s)}{d_F(s)}$$
We are interested in the closed-loop transfer functions, expressed similarly to the previous problem (ie., simple fractions involving the individual $n$ and $d$ polynomials)

   (a) What is the transfer function from $r \to u$

   (b) What is the transfer function from $d \to y$

   (c) What is the transfer function from $r \to y$

   (d) What is the closed-loop characteristic polynomial of the system

8. The governing equations for a DC motor are
$$V(t) - RI(t) - K\omega(t) = 0, \qquad J\dot{\omega}(t) = KI(t) - \alpha\omega(t) + T(t)$$

   - $V(t)$ is the voltage across the motor winding, at the terminals; $I(t)$ is the current flowing through the motor windings; $\omega(t)$ is the angular velocity of the shaft; and $T(t)$ is the sum of all torques applied externally to the motor shaft (interactions with other linkages/inertias, disturbances, etc)
   - $J, K, \alpha, R$ are **positive** constants, which are properties of the motor itself, namely the shaft/gear inertia, the motor-constant, the viscous friction coefficient of the bearings, and the electrical resistance in the windings.

   Consider two behaviors of the motor

   **shorted:** the terminals are connected together so that $V(t) = 0$ for all $t$

   **open:** the terminals are not connected to anything or to each other, so $I(t) = 0$ for all $t$

   (a) Treating $T$ as the single input, and $(\omega, I)$ as the two outputs, write state equations for the *shorted* system (hint: there is 1 state, and in this case, 1 input, and 2 outputs)

   (b) For the *shorted* system, what is the time-constant?

   (c) For the *shorted* system, what is the transfer-function from $T$ to $\omega$?

   (d) For the *shorted* system, what is the steady-state gain from $T$ to $\omega$?

(e) Treating $T$ as the single input, and $(\omega, V)$ as the 2 outputs, write state equations for the *open* system

(f) For the *open* system, what is the transfer-function from $T$ to $\omega$?

(g) For the *open* system, what is the steady-state gain from $T$ to $\omega$?

(h) For the *open* system, what is the time-constant?

(i) If you, with your hands/fingers, apply a torque to the two separate systems, which will be easier to turn? Why?

9. Consider the plant/controller pair (for the standard $(P, C)$ configuration described on the front page), described in transfer-function form

$$P(s) = \frac{1}{s-\alpha}, \qquad C = \sqrt{5} \cdot \alpha \ \ (\approx 2.24\alpha)$$

where $\alpha > 0$ is a constant. Note that $C$ is a constant gain. **Remark:** For the questions below, if you have trouble, first work out the answers for the case $\alpha = 1$, then go back and see how to generalize to an arbitrary, fixed, positive $\alpha$.

(a) Is the plant $P$ a stable system? Is the closed-loop system stable?

(b) Since the closed-loop system is a first-order system, what is the time-constant of the closed-loop system. Your answer will be in terms of $\alpha$.

(c) Define $L(s) := P(s)C$. Using mathematical manipulations, find the frequency $\omega_c \geq 0$ such that $|L(j\omega_c)| = 1$. Your answer will be in terms of $\alpha$.

(d) What is $\angle L(j\omega_c)$, in radians? **Hint:** Your answer **will not** depend on $\alpha$.

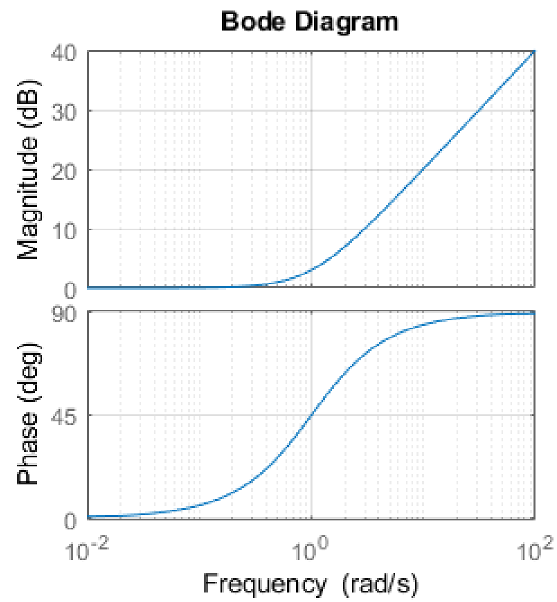
(e) Make a simple sketch, in the complex-plane $\mathbb{C}$, showing the unit-circle, and mark the value of $L(j\omega_c)$.

(f) What is the time-delay margin (in terms of $\alpha$) for the closed-loop system?

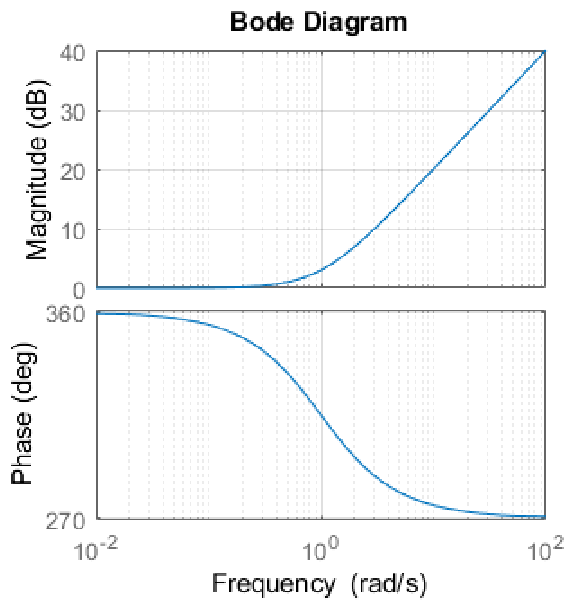
10. The Bode plots (Magnitude and Phase) for 4 simple transfer functions,

$$G_1(s) = \frac{1}{-s+1}, \qquad G_2(s) = -s+1, \qquad G_3(s) = \frac{1}{s+1}, \qquad G_4(s) = s+1$$

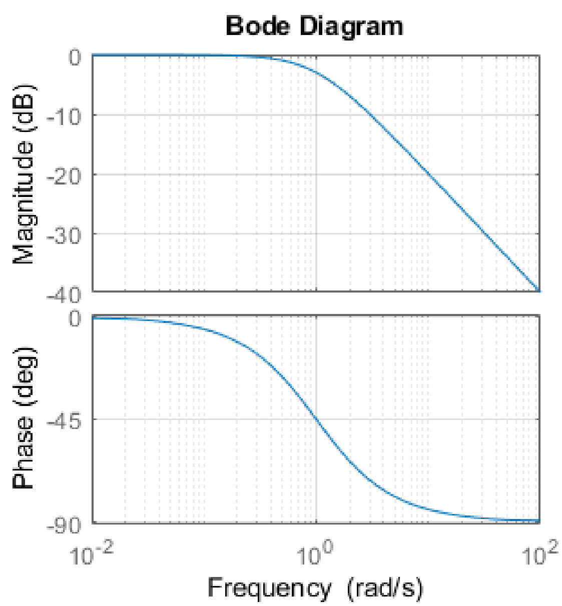
are shown below. Match each system to its corresponding plot



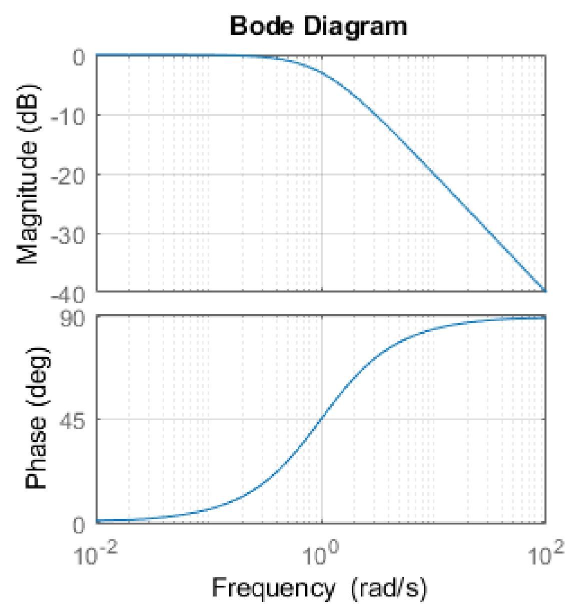
Bode Diagram
Magnitude (dB)
40
30
20
10
0
Phase (deg)
90
45
0
$10^{-2}$
$10^{0}$
$10^{2}$
Frequency (rad/s)

Bode Diagram
Magnitude (dB)
40
30
20
10
0
Phase (deg)
360
270
$10^{-2}$
$10^{0}$
$10^{2}$
Frequency (rad/s)

Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
Phase (deg)
0
-45
-90
$10^{-2}$
$10^{0}$
$10^{2}$
Frequency (rad/s)

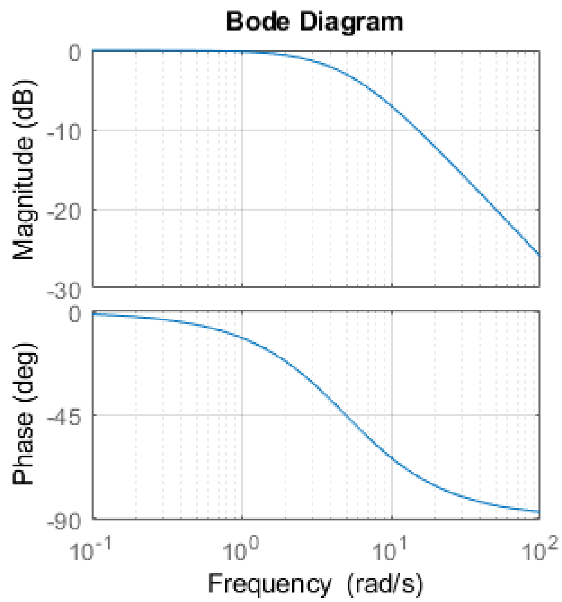



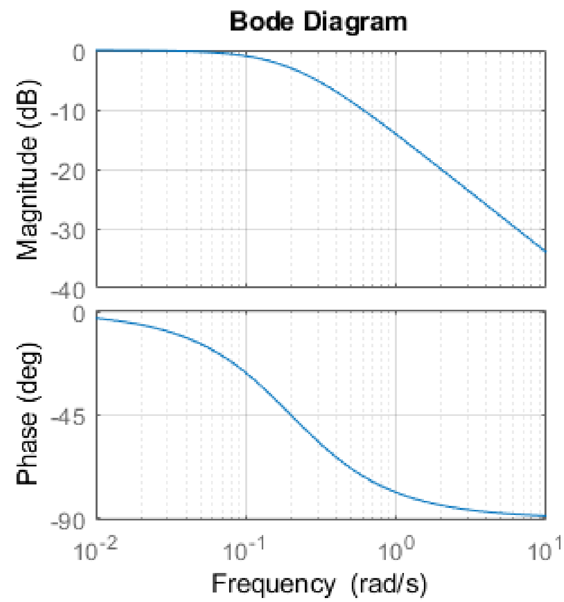
Bode Diagram
Magnitude (dB)
0
-10
-20
-30
-40
Phase (deg)
90
45
0
$10^{-2}$
$10^{0}$
$10^{2}$
Frequency (rad/s)

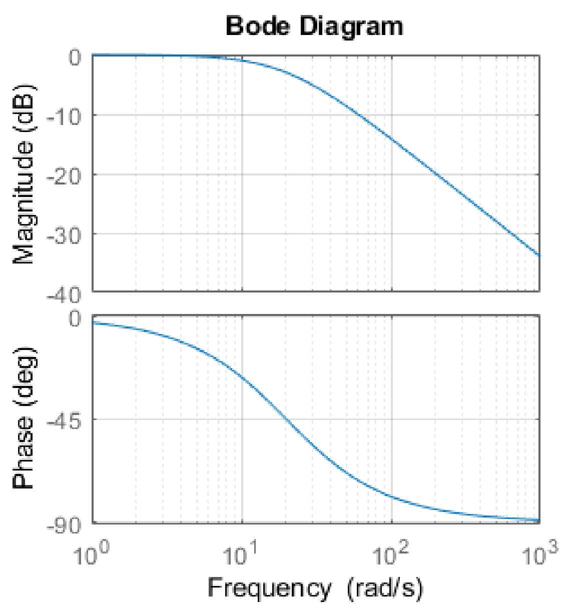
11. The Bode plots (Magnitude and Phase) for 4 simple transfer functions,

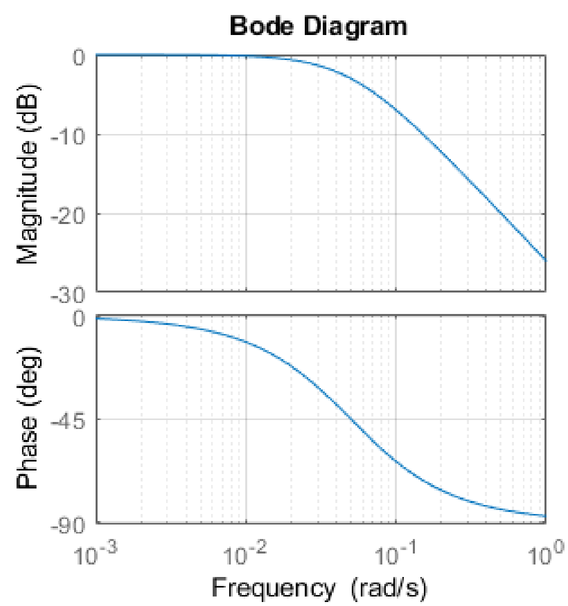
$$G_1(s) = \frac{20}{s+20}, \qquad G_2(s) = \frac{0.2}{s+0.2}, \qquad G_3(s) = \frac{0.05}{s+0.05}, \qquad G_4(s) = \frac{5}{s+5}$$

are shown below. Match each system to its corresponding plot


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

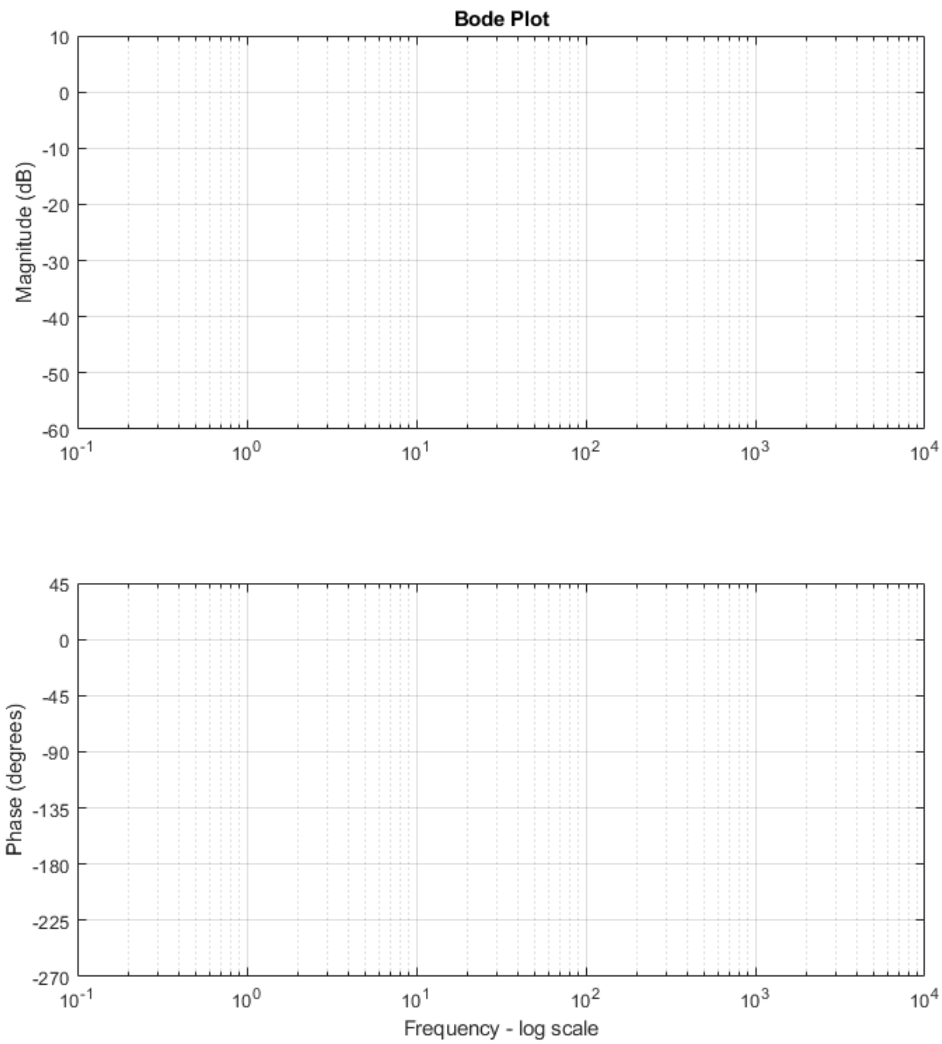

Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

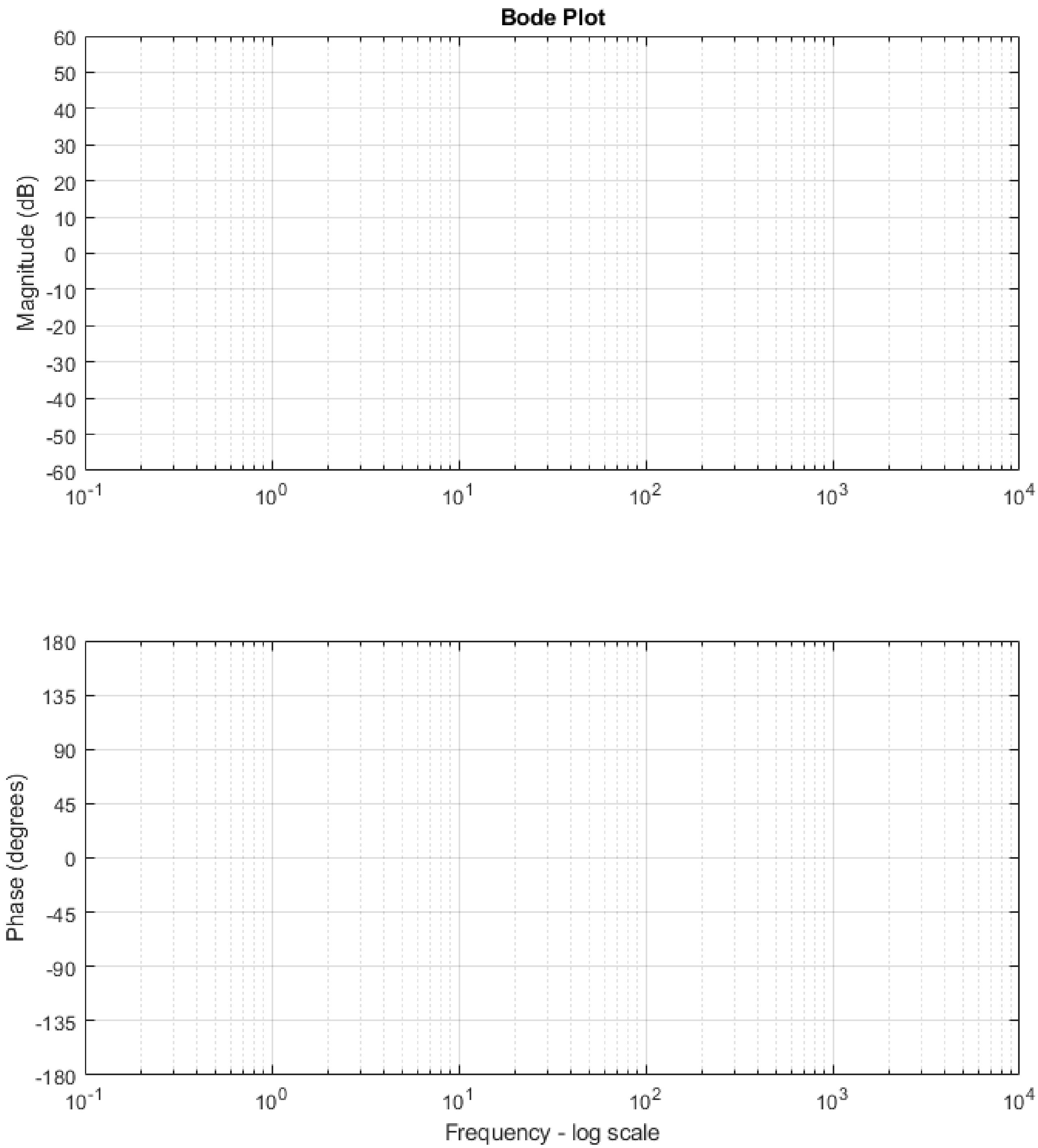

Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)

12. Make a straight-line Bode plot (magnitude and phase) for the transfer function

$$G(s) = 9\frac{(-s+100)(s+1)}{(s+3)(s+10)(s+30)}$$

on the graph paper below. Be sure to normalize the terms properly (ie., write $(s+30)$ as $(\frac{s}{30}+1)$, and account for all the accumulated scaling factors at the end with a single gain adjustment). **Important:** The axes-limits on this graph paper are appropriate for the final result. Use the graph-paper on the next page (which has wider limits, and easier to get started with) to work out your solution, then transfer it to this page.




Bode Plot
Magnitude (dB)
60
50
40
30
20
10
0
-10
-20
-30
-40
-50
-60
$10^{-1}$
$10^{0}$
$10^{1}$
$10^{2}$
$10^{3}$
$10^{4}$
Phase (degrees)
180
135
90
45
0
-45
-90
-135
-180
$10^{-1}$
$10^{0}$
$10^{1}$
$10^{2}$
$10^{3}$
$10^{4}$
Frequency - log scale

13. State whether each statement is True or False. No reasons need be given...

    (a) For all $A_1, A_2 \in \mathbb{R}^{n\times n}$, and all $t \geq 0$, $e^{(A_1+A_2)t} = e^{A_1 t}e^{A_2 t}$

    (b) For all $A \in \mathbb{R}^{n\times n}$, and all $t \in \mathbb{R}$, $e^{(-A)t} = e^{A(-t)}$

    (c) For all $A \in \mathbb{R}^{n\times n}$, and all $t \geq 0$, $e^{At}e^{-At} = I_n$

    (d) For all $A \in \mathbb{R}^{n\times n}$, and all $t \in \mathbb{R}$, the matrix $e^{At}$ is invertible.

    (e) For all $A \in \mathbb{R}^{n\times n}$, and all $t_1, t_2 \in \mathbb{R}$, $e^{A(t_1+t_2)} = e^{At_2}e^{At_1}$

    (f) For all $A \in \mathbb{R}^{n\times n}$, and all $t \in \mathbb{R}$, $Ae^{At} = e^{At}A$

    (g) For all $A_1, A_2 \in \mathbb{R}^{n\times n}$, and all $t \geq 0$, $A_1 e^{A_2 t} = e^{A_1 t}A_2$

    (h) If $A \in \mathbb{R}^{n\times n}$ is invertible, then for all $t \in \mathbb{R}$, $A^{-1}e^{At} = e^{At}A^{-1}$

    (i) If $A \in \mathbb{R}^{n\times n}$, then for all $t \in \mathbb{R}$, and all $\omega \in \mathbb{R}$, $(j\omega I - A)e^{At} = e^{At}(j\omega I - A)$

    (j) If $A \in \mathbb{R}^{n\times n}$ has no imaginary eigenvalues, then for all $t \in \mathbb{R}$, and all $\omega \in \mathbb{R}$,
    $$(j\omega I - A)^{-1}e^{At} = e^{At}(j\omega I - A)^{-1}$$

    (k) Consider the two expressions
    $$E_1 := (I + tA + \frac{t^2}{2}A^2)(I + tB + \frac{t^2}{2}B^2), \qquad E_2 := I + t(A+B) + \frac{t^2}{2}(A+B)^2$$
    where $A, B \in \mathbb{R}^{n\times n}$. Expand both expressions, and obtain the coefficient-matrix associated with the $t^2$ term. Are they equal, in general?

    (l) Which disequality (circle your answer) comes from the reasoning in the previous problem
    $$e^{(A-B)t} \neq e^{At}e^{-Bt}, \qquad Be^{At} \neq e^{Bt}A, \qquad e^{ABt} \neq e^{At}e^{Bt}$$

14. A 2nd-order system, which represents the dynamical equations for a feedback controller, has state equations

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} (r(t) - y_{\text{meas}}(t))$$

and output equation

$$u(t) = \begin{bmatrix} 7 & 8 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + 9(r(t) - y_{\text{meas}}(t))$$

The numbers are not necessarily realistic, but that is not the focus of the problem. Using the same ideas as in the lab, this control "strategy" is to implemented at a sample-rate of 10 milliseconds (0.01 seconds).

Accessing the reference $r$ and the measurement $y$ is accomplished with 2 external functions that we are free to call as needed:

- The reference signal is computed by an externally defined function named `getReference`. That function has one input argument (time, as an integer, in units of milliseconds), and it returns a `float`.
- At any time, the process output, $y$ can be measured and its value returned with the function `makeMeasurement`. This function has no input arguments, and returns a `float`.

The control action, $u(t)$ can be "sent" to the actuator with the command `setControlValue`. This function has one input argument, the value of the control action (as a `float`). The behavior of the function is that the control signal immediately gets set to the specified value, and is held constant (at this value), until the function is called again.

On the next page, fill in the blank lines of the code to complete this program.

```
int T1Val, ExpLength = 30000, SampleRate = _____ ;
float rVal, yVal, eVal, uVal;
float z1, z2, z1Dot, z2Dot;
z1 = 0.0;
z2 = 0.0;
float SampleTime;
SampleTime = 0.001*SampleRate;
clearTimer(T1);
clearTimer(T2);
// Begin control at t=0
setControlValue(9*(getReference(0)-makeMeasurement));

while (time1[T1] < ExpLength) {
     T1Val = time1[T1];

     if (time1[T2]>= ______________ ) {

         clearTimer(_______);

         ______ = makeMeasurement;

         rVal = ________________________;

         _______ = rVal - yVal;

         uVal = _______________________________________;

         setControlValue(_______________);

         z1Dot = ___________________________________________________________;

         z2Dot = ___________________________________________________________;

         z1 = ___________  +  _________________________________;

         z2 = ___________  +  _________________________________;
     }
}
```

15. Let $\sigma, \beta \in \mathbb{R}$, with $\sigma^2 + \beta^2 > 0$ (in other words, at least one of them is nonzero).

Consider

$$A := \begin{bmatrix} \sigma & \beta \\ -\beta & \sigma \end{bmatrix}$$

(a) What are the eigenvalues of $A$

(b) Is $A$ invertible?

(c) What is the inverse of $A$

(d) $A$ is a special matrix we studied in class. What is $e^{At}$

(e) What is $(sI_2 - A)^{-1}$

(f) What is $A^{-1}\left(e^{At} - I_n\right)$

16. Consider the linear system $\dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t)$ for

$$A := \begin{bmatrix} -4 & 3 \\ -3 & -4 \end{bmatrix}, \qquad B := \begin{bmatrix} -2 \\ 3 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

(a) What is the transfer function from $u$ to $y$

(b) What is the exact expression for the response $y(t)$ due to a unit-step input, $u(t) = 1$ for all $t$, starting from initial condition $x(0) = 0$. See problem 15, and **Facts** on front page if needed.