First Midterm Examination
Closed Books and Closed Notes
Answer Two Questions of your Choice

Question 1

A Stick-Slip Problem (25 POINTS)

As shown in Figure 1, a particle of mass m is free to move on the rough outer surface of a spinning cone. The particle is subject to a vertical gravitational force. The coefficients of static friction and dynamic friction between the surface of the cone and the particle are denoted by μ_s and μ_k, respectively. The position vector of the particle has the representation

$$\mathbf{r} = (r_0 + s\cos(\alpha))\,\mathbf{e}_r - s\sin(\alpha)\,\mathbf{E}_z, \tag{1}$$

where s has dimensions of length.

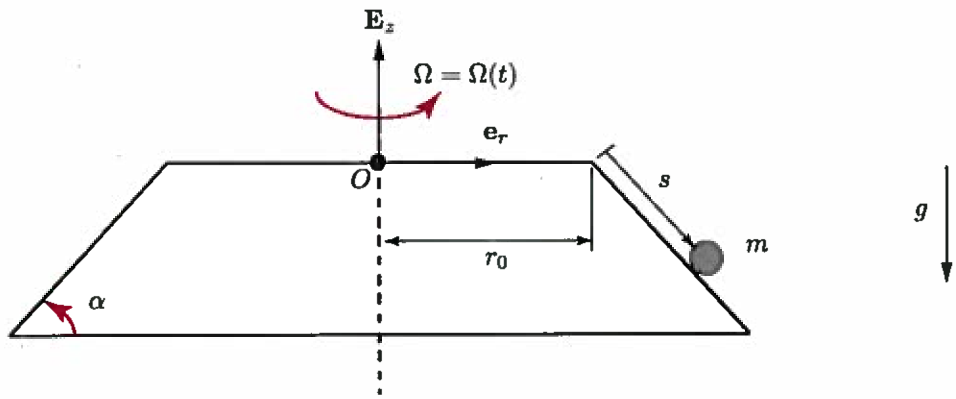


Figure 1: *Schematic of a particle of mass m that is free to move on the rough outer surface of a spinning cone. The coordinate s and dimension r_0 are also defined in this figure.*

(a) *(5 Points)* Assume that the particle is moving on the surface of the cone. Then, starting from (1), establish expressions for the velocity vector $\mathbf{v}$ and acceleration vector $\mathbf{a}$ of the particle.

(b) *(2 Points)* If the cone is being spun with an angular speed $\Omega = \Omega(t)$, then what is the relative velocity vector $\mathbf{v}_{\text{rel}}$ of the particle?

(c) *(5 Points)* Draw a freebody diagram of the particle. Your freebody diagram should include clear expressions for the friction force. Both cases of static and dynamic friction should be represented.

(d) *(8 Points)* Suppose that the particle is stationary on the cone: $s = s_0$. Show that the particle will remain in this state if the following inequality is satisfied:

$$\mu_s\,|g - ?| \geq \sqrt{(??)^2 + \left(???\dot{\Omega}\right)^2}. \tag{2}$$

For full credit supply, the missing terms. Why is it also necessary for $\Omega^2 < \frac{g\cos(\alpha)}{(r_0 + s_0\cos(\alpha))\sin(\alpha)}$?

(e) *(5 Points)* With the help of the work-energy theorem $\dot{E} = \mathbf{F}_{nc}\cdot\mathbf{v}$, show that the energy of the particle in **(d)** is changing at the following rate:

$$\dot{E} = \frac{d}{dt}\left(\frac{m\,(r_0 + s_0\cos(\alpha))^2}{2}\Omega^2\right). \tag{3}$$

Question 2

A Simple Suspension System (25 POINTS)

As shown in Figure 2, a particle of mass m is attached to a fixed point O by a massless rod of length ℓ. In addition to a vertical gravitational force $mg\mathbf{E}_x$, a spring force acts on the particle. This spring force is supplied by a spring of unstretched length ℓ_0 and stiffness K.

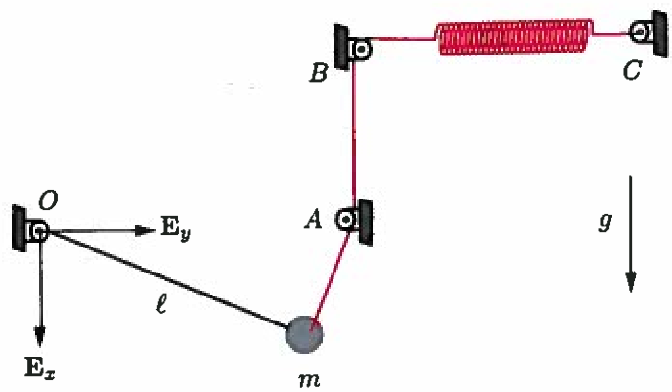


Figure 2: *A simple pendulum that is restrained by a spring force.*

(a) *(5 Points)* Starting from the representation

$$\mathbf{r} = \ell \mathbf{e}_r, \tag{4}$$

establish representations for the speed v, velocity vector $\mathbf{v}$, acceleration vector $\mathbf{a}$, and the unit tangent vector $\mathbf{e}_t$ to the path of the particle.

(b) *(2 Points)* If $\mathbf{r}_A \approx \ell \mathbf{E}_y$ and the spring is unstretched when $\theta = 90^o$, then show that

$$\ell_0 = \|\mathbf{r}_B - \mathbf{r}_C\| + \|\mathbf{r}_A - \mathbf{r}_B\|, \qquad \epsilon = \|\mathbf{r} - \mathbf{r}_A\|. \tag{5}$$

(c) *(5 Points)* Draw a freebody diagram of the particle. For full credit provide a clear expression for the spring force.

(d) *(5 Points)* Show that the differential equations governing the motion of the particle is

$$\ddot{\theta} = ?? - \frac{g}{\ell}\sin(\theta). \tag{6}$$

For full credit supply, the missing term.

(e) *(3 Points)* Suppose that the mass particle is instantaneously at rest with $\theta = \arctan\left(\frac{K\ell}{mg}\right)$. Discuss the ensuing motion of the particle.

(f) *(5 Points)* Starting from the work-energy theorem $\dot{E} = \mathbf{F}_{nc} \cdot \mathbf{v}$, prove that the total energy E of the particle is conserved. For full credit, provide an expression for E.

Question 3
A Particle on a Space Curve (25 POINTS)

As shown in Figure 3, a bead of mass m is free to move on a smooth rail. The bead is subject to a vertical gravitational force. The rail has the shape of a spiral:

$$\mathbf{r} = f(\theta)\,\mathbf{e}_r, \tag{7}$$

where f is a known function of θ.

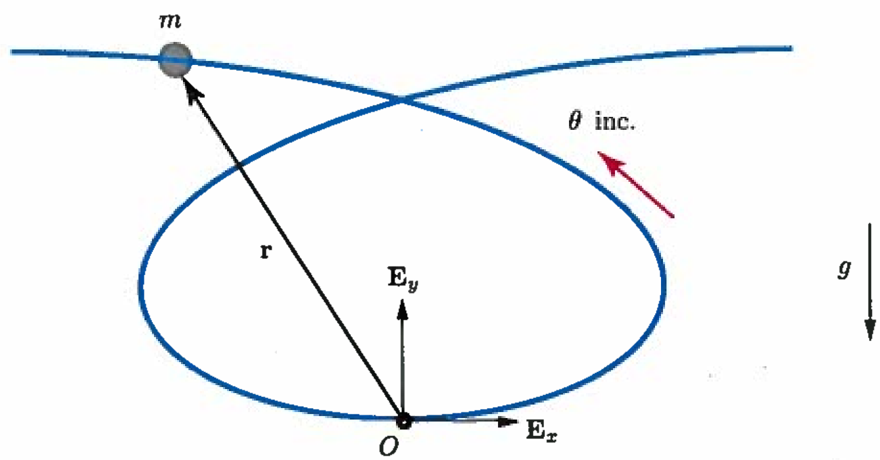


Figure 3: *Schematic of a particle of mass m which is free to move on a smooth rail that has a spiral shape.*

(a) *(8 Points)* Assume that the particle is moving on the rail. Starting from $\mathbf{r} = f(\theta)\,\mathbf{e}_r$, establish expressions for the velocity vector $\mathbf{v}$, speed v of the particle, and total energy E of the particle.

(b) *(5 Points)* Assuming that $\dot{\theta} > 0$ and $\mathbf{e}_b = \mathbf{E}_z$. Verify that

$$\sqrt{f^2 + f'f'}\,\mathbf{e}_t = f\mathbf{e}_\theta + f'\mathbf{e}_r, \qquad \sqrt{f^2 + f'f'}\,\mathbf{e}_n = -f\mathbf{e}_r + f'\mathbf{e}_\theta, \tag{8}$$

where $f' = \frac{df}{d\theta}$.

(c) *(3 Points)* Draw a freebody diagram of the particle.

(d) *(7+2 Points)* Suppose that the curve is an Archimedean spiral. The respective functions f and κ for this curve are

$$f(\theta) = a\theta, \qquad \kappa = \frac{2+\theta^2}{a\left(\sqrt{1+\theta^2}\right)^3}, \tag{9}$$

where a is a positive constant.

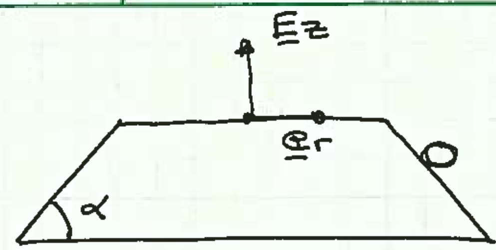
(i) Show that the normal force acting on the particle is

$$\mathbf{N} = \frac{m}{\sqrt{1+\theta^2}}\left(??\dot{\theta}^2 + g?\right)\mathbf{e}_n. \tag{10}$$

For full credit supply, the missing terms.

(ii) How does your answer for $\mathbf{N}$ in (i) simplify when the particle is instantaneously located at the origin?

QUESTION 1



(a) $\underline{r} = (r_0 + s\cos(\alpha))\underline{e}_r - s\sin(\alpha)\underline{E}_z$

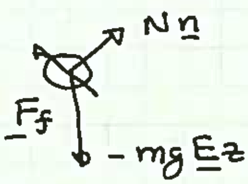
$$\underline{v} = \dot{s}(\cos\alpha\,\underline{e}_r - \sin(\alpha)\underline{E}_z) + (r_0 + s\cos\alpha)\dot{\theta}\,\underline{e}_\theta$$

$$\underline{a} = \ddot{s}(\cos\alpha\,\underline{e}_r \mp \sin(\alpha)\underline{E}_z) + 2\dot{s}\dot{\theta}\cos\alpha\,\underline{e}_\theta + (r_0 + s\cos\alpha)\ddot{\theta}\,\underline{e}_\theta - (r_0 + s\cos\alpha)\dot{\theta}^2\underline{e}_r$$

Note that $\dot{\theta} \neq \Omega$ unless particle is stuck on surface

(b) $$\underline{v}_{rel} = \underline{v} - (r_0 + s\cos\alpha)\Omega\,\underline{e}_\theta$$
$$= \dot{s}(\cos\alpha\,\underline{e}_r - \sin(\alpha)\underline{E}_z) + (r_0 - s\cos\alpha)(\dot{\theta} - \Omega)\,\underline{e}_\theta$$

(c)



$$\underline{F}_f = \begin{cases} F_{f_1}\underline{t}_1 + F_{f_2}\underline{t}_2 & \text{(static)} \\ -\mu_k\|\underline{N}\|\dfrac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|} & \text{(dynamic)} \end{cases}$$

$$\underline{t}_1 = \cos(\alpha)\underline{e}_r - \sin(\alpha)\underline{E}_z, \quad \underline{t}_2 = \underline{e}_\theta$$
$$\underline{n} = \underline{t}_1 \times \underline{t}_2 = \cos\alpha\,\underline{E}_z + \sin(\alpha)\underline{e}_r$$

(d) $$\underline{F} = m\underline{a} = m(r_0 + s_0\cos\alpha)\dot{\Omega}\,\underline{e}_\theta - m(r_0 + s_0\cos\alpha)\Omega^2\underline{e}_r$$

Easiest to take components wrt $\{\underline{t}_1, \underline{t}_2, \underline{n}\}$ to solve for $\underline{F}_f$ and $\underline{N}$:

- $\cdot\,\underline{t}_1$: $F_{f_1} + mg\sin\alpha = -m(r_0 + s_0\cos\alpha)\Omega^2\cos\alpha$] Solve for F_{f_1}
- $\cdot\,\underline{t}_2$: $F_{f_2} = m(r_0 + s_0\cos\alpha)\dot{\Omega}$] Solve for F_{f_2}
- $\cdot\,\underline{n}$: $N - mg\cos\alpha = -m(r_0 + s_0\cos\alpha)\Omega^2\sin\alpha$] Solve for N

Static friction criterion $\|\underline{F}_f\| \le \mu_s\|\underline{N}\|$

We also need $N = \underline{N}\cdot\underline{n} > 0$ for particle to stay on surface

Substituting into the static friction criterion and dividing by m:

$$\mu_s \left| g\cos\alpha - (r_0 + s_0\cos\alpha)\Omega^2\sin\alpha \right|$$

$$\geq \sqrt{\left(\frac{F_{f_1}}{m} = -g\sin\alpha - (r_0 + s_0\cos\alpha)\Omega^2\cos\alpha\right)^2 + \left(\frac{F_{f_2}}{m} = (r_0+s_0\cos\alpha)\dot{\Omega}\right)^2}$$

For the particle to stay on surface: $N > 0$

Hence $\quad mg\cos\alpha - m(r_0 + s_0\cos\alpha)\Omega^2\sin\alpha > 0$

$$\Rightarrow \quad \Omega^2 < \frac{g\cos\alpha}{(r_0 + s_0\cos\alpha)\sin\alpha}$$

(e) $\quad \dot{E} = \underline{F}_{nc}\cdot\underline{v} = \underline{F}_f\cdot\underline{v} + \underline{N}\cdot\underline{v}$

$$= \underline{F}_f\cdot\underline{v} + 0$$

$$= F_{f_2}(r_0 + s_0\cos\alpha)\Omega$$

$$= m(r_0 + s_0\cos\alpha)^2\Omega\dot{\Omega}$$

$$= \frac{d}{dt}\left(\frac{m}{2}(r_0 + s_0\cos(\alpha))^2\Omega^2\right)$$

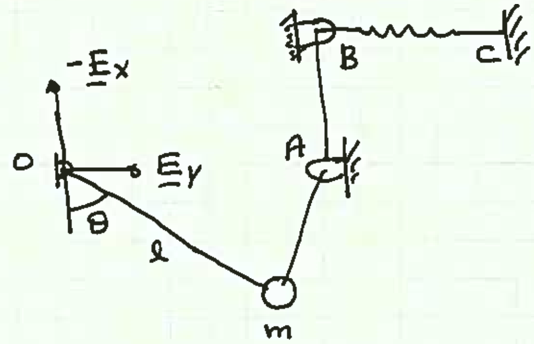
We can also find this result by noting that when the particle is stuck on the surface
$\underline{v}_{rel} = \underline{0} \qquad \underline{v} = (r_0 + s_0\cos\alpha)\Omega\underline{e}_\theta$

Thus
$$E = T + U = \frac{1}{2}m\underline{v}\cdot\underline{v} + mgz$$

$$= \frac{1}{2}m(r_0 + s_0\cos\alpha)^2\Omega^2 + mgs_0\sin\alpha$$

Hence $\quad \dot{E} = \frac{d}{dt}\left(\frac{m}{2}(r_0 + s_0\cos\alpha)^2\Omega^2\right)$

QUESTION 2



(a) $\underline{r} = \ell \underline{e}_r$

$\underline{v} = \ell \dot{\theta} \underline{e}_\theta$

$\underline{a} = \ell \ddot{\theta} \underline{e}_\theta - \ell \dot{\theta}^2 \underline{e}_r$

$v = |\ell \dot{\theta}| \qquad \underline{e}_t = \dfrac{\dot{\theta}}{|\dot{\theta}|} \underline{e}_\theta$

(b) Unstretched length ℓ_0 reached when $\underline{r} = \underline{r}_A$ then

$$\ell_0 = \|\underline{r}_A - \underline{r}_B\| + \|\underline{r}_C - \underline{r}_B\|$$

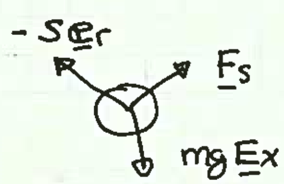
Stretched length of the spring is

$$\ell = \|\underline{r}_A - \underline{r}_B\| + \|\underline{r}_C - \underline{r}_B\| + \|\underline{r} - \underline{r}_A\|$$

Hence

$$\varepsilon = \ell - \ell_0 = \|\underline{r} - \underline{r}_A\| \quad = \text{extension of spring}$$

(c)



$$\underline{F}_s = -K\varepsilon \frac{\underline{r} - \underline{r}_A}{\|\underline{r} - \underline{r}_A\|} = -K(\underline{r} - \underline{r}_A)$$

$$\underline{r} - \underline{r}_A = \ell \underline{e}_r - \ell \underline{E}_y$$

(d) $\underline{F} = m\underline{a}: \qquad -S\underline{e}_r + \underline{F}_s + mg\underline{E}_x = m\ell\ddot{\theta}\underline{e}_\theta - \ell\dot{\theta}^2\underline{e}_r$

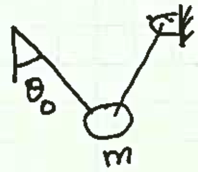
To get EOM we consider $\underline{e}_\theta$ component of this equation:

$$m\ell\ddot{\theta} = -K(\underline{r} - \underline{r}_A)\cdot\underline{e}_\theta + mg\underline{E}_x\cdot\underline{e}_\theta$$

$$= +K\cos\theta\,\ell - mg\sin\theta$$

$$\Rightarrow \quad \ddot{\theta} = \frac{K}{m}\cos\theta - \frac{g}{\ell}\sin\theta$$

(e) If $\theta = \theta_0$ where $\tan\theta_0 = \dfrac{Kl}{mg}$



then $mg\sin\theta_0 = Kl\cos\theta_0$ (Gravity balances spring force)

and from Equation of motion $\ddot{\theta} = 0$

Thus $\dot{\theta}$ stays at 0 and θ stays @ θ_0.

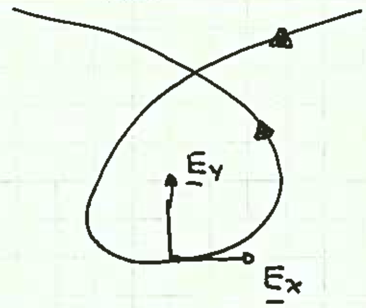
So particle remains at this location forever

(f) $$E = \tfrac{1}{2} m l^2 \dot{\theta}^2 + \tfrac{1}{2} K \varepsilon^2 - mgl\cos\theta$$

$$\dot{E} = \underline{F}_{nc} \cdot \underline{v} = -S\underline{e}_r \cdot \underline{v} = -S\underline{e}_r \cdot l\dot{\theta}\underline{e}_\theta = 0.$$

Hence $\dot{E} = 0$ so E is conserved.

$\boxed{\text{QUESTION 3}}$



(a) $\underline{r} = f(\theta)\underline{e}_r$

$$\underline{v} = \dot{\theta} f' \underline{e}_r + f \dot{\theta} \underline{e}_\theta$$

$$\underline{a} = \ddot{\theta}(f'\underline{e}_r + f\underline{e}_\theta) + \dot{\theta}^2 f'' \underline{e}_r + \dot{\theta}^2 f' \underline{e}_\theta + f'\dot{\theta}^2 \underline{e}_\theta - f\dot{\theta}^2 \underline{e}_r$$

$$= \ddot{\theta}(f'\underline{e}_r + f\underline{e}_\theta) + \dot{\theta}^2(f''\underline{e}_r - f\underline{e}_r) + \dot{\theta}^2(2f')\underline{e}_\theta$$

$$v = \dot{\theta}\sqrt{f^2 + f'f'}$$

$$E = \tfrac{1}{2}mv^2 + mg\,\underline{E}_y \cdot \underline{r}$$

$$= \tfrac{1}{2}m\dot{\theta}^2(f^2 + f'f') + f mg \sin\theta$$

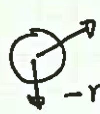
$$\boxed{\begin{aligned} \dot{f} &= \frac{df}{dt} = \frac{df}{d\theta}\dot{\theta} \\ \ddot{f} &= \frac{d^2f}{d\theta^2}\dot{\theta}^2 + \frac{df}{d\theta}\ddot{\theta} \end{aligned}}$$

(b) $\underline{v} = v\underline{e}_t \Rightarrow \underline{e}_t = \dfrac{\underline{v}}{v} = \dfrac{1}{\sqrt{ff + f'f'}}\left(f'\underline{e}_r + f\underline{e}_\theta\right)$

Now $\underline{e}_n = \underline{e}_b \times \underline{e}_t = \underline{E}_z \times \underline{e}_t$

$$= \frac{1}{\sqrt{ff + f'f'}}\left(f'\underline{e}_\theta - f\underline{e}_r\right)$$

(c)



$\underline{N} = N_n\underline{e}_n + N_b\underline{e}_b$

(d) $\underline{F} = m\underline{a}$: $\quad m\dot{v}\underline{e}_t + m\kappa v^2\underline{e}_n = -mg\,\underline{E}_y + \underline{N}$

$\cdot\,\underline{e}_t \qquad m\dot{v} = -mg\,\underline{E}_y \cdot \underline{e}_t$

$\cdot\,\underline{e}_n \qquad m\kappa v^2 = N_n - mg\,\underline{E}_y \cdot \underline{e}_n$

$\cdot\,\underline{e}_b \qquad 0 = N_b$

(i)

Hence $\underline{N} = N_n\underline{e}_n = \left(m\kappa v^2 + mg\,\underline{E}_y \cdot \underline{e}_n\right)\underline{e}_n$

Now $$\underline{E}_y \cdot \underline{e}_n = \frac{1}{a\sqrt{1+\theta^2}}\left(-a\theta\,\underline{e}_r + a\,\underline{e}_\theta\right)\cdot \underline{E}_y$$

$$= \frac{1}{a\sqrt{1+\theta^2}}\left(-a\theta\sin\theta + a\cos\theta\right)$$

$$= \frac{1}{\sqrt{1+\theta^2}}\left(\cos\theta - \theta\sin\theta\right)$$

$$Kv^2 = \dot{\theta}^2(1+\theta^2)a^2\left(\frac{2+\theta^2}{a(1+\theta^2)\sqrt{1+\theta^2}}\right)$$

$$= \frac{2+\theta^2}{\sqrt{1+\theta^2}}\,a\dot{\theta}^2$$

$$\underline{N} = \frac{m}{\sqrt{1+\theta^2}}\left((2+\theta^2)a\dot{\theta}^2 + g(\cos\theta - \theta\sin\theta)\right)\underline{e}_n$$

(iii)

as particle passes origin: $\theta = 0$ $\quad \underline{e}_n = \underline{E}_y$, $K = \frac{2}{a}$, $v = a\dot{\theta}$

$$\underline{N} = m\left(2a\dot{\theta}^2 + g\right)\underline{E}_y$$

as expected. Normal force balances gravity and centripetal acceleration.