Second Midterm Exam Solutions

Place all answers on the question sheet provided. The exam is open textbook and open notes/handouts/homework. You are allowed to use a calculator, but not a computer, tablet or smartphone. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument. This exam has a total of 50 points.

First Name: ______________________________

Last Name: ______________________________

1(a)	1(b)	1(c)	1(d)	2(a)	2(b)	2(c)	Total

Honor Code

I resolve

i) not to give or receive aid during this examination, and

ii) to take an active part in seeing that other students uphold this Honor Code.

Signature: ______________________________

1. George is known to always be able to help people with their computer problems. His friends and work-colleagues request his help at times that follow a Poisson process with rate $1/24$ per hour, i.e. one per day, and each time he helps somebody he has to stop his own work for a uniform amount of time between $[1/2, 2]$, i.e., between 30 min to two hours. The only times when George says no to a request is when he's helping somebody else.

   We model George's day as an on-off renewal process where the on period refers to the time he spends helping somebody, and the off period when he's not helping anybody.

   (a) [6 PTS] Compute the expected length of both the on and off periods.

   Solution:

   The on period has a Uniform$(1/2, 2)$ distribution, and therefore

$$E[\text{On period}] = \frac{2 + 1/2}{2} = \frac{5}{4},$$

   and since after completing a request, the next one to arrive will do so in an exponential amount of time with rate $1/24$, then

$$E[\text{Off period}] = 24.$$

(b) [7 PTS] Compute the long-run proportion of time that George spends helping other people.

Solution:

$$\frac{E[\text{On period}]}{E[\text{On period}] + E[\text{Off period}]} = \frac{5/4}{5/4 + 24} = \frac{5}{101}$$

(c) [8 PTS] Compute the expected number of times George says no to somebody asking for his help during an on period.

Solution:

The number of people who will ask George for help during a period of length t is a Poisson random variable $N(t)$ with mean $t/24$, and so if we let U be a Uniform$[1/2, 2]$ random variable, representing the amount of time George is busy helping somebody else, then

$$\begin{aligned} E[\text{Number of times he says no}] &= E[N(U)] = E[E[N(U)|U]] = E[U/24] = E[U]/24 \\ &= \frac{5}{4(24)} = \frac{5}{96} \end{aligned}$$

(d) [8 PTS] Compute the variance of the number of times George says no to somebody asking for his help during an on period.

Solution:

We are interested in the variance of $N(U)$, where $N(t)$ and U are the same from part (c). Therefore, using the formula of the total variance,

$$\begin{aligned}\text{var}(N(U)) &= E[\text{var}(N(U)|U)] + \text{var}(E[N(U)|U]) = E[U/24] + \text{var}(U/24) \\ &= \frac{E[U]}{24} + \frac{\text{var}(U)}{(24)^2}\end{aligned}$$

We already computed $E[U] = 5/4$, and the variance of a Uniform$(1/2, 2)$ is

$$\text{var}(U) = \frac{(2-1/2)^2}{12} = \frac{(3/2)^2}{12} = \frac{3}{16}$$

Hence,

$$\text{var}(N(U)) = \frac{5}{4(24)} + \frac{3}{16(24)^2} = \frac{1}{24}\left(\frac{5}{4} + \frac{1}{16(8)}\right) = \frac{161}{16(8)(24)} = \frac{161}{3072}$$

2. Joyce receives Facebook updates from her three closest friends: Amy, Brenda and Cindy. Each of her friends has different Facebook habits, with Amy sending her messages at a rate of 1/7 per day, Brenda at a rate of 1/2 per day, and Cindy at a rate of 1 per day. We assume that the number of messages Joyce receives from her friends are independent Poisson processes.

   (a) [5 PTS] Let $N(t)$ denote the number of messages Joyce receives from her three friends during the period $[0, t]$. Write down the PMF of $N(t)$, i.e., $P(N(t) = n)$ for $n = 0, 1, 2, \ldots$.

   Solution:

   Since the superposition of independent Poisson processes is again a Poisson process whose rate is the sum of the rates, then $N(t)$ is a Poisson process with rate

$$\frac{1}{7} + \frac{1}{2} + 1 = \frac{2 + 7 + 14}{14} = \frac{23}{14}$$

   or equivalently,

$$P(N(t) = n) = \frac{e^{-23t/14}(23t/14)^n}{n!}, \ldots n = 0, 1, 2, \ldots.$$

(b) [8 PTS] Compute the probability that the next Facebook update Joyce receives from her three friends comes from Cindy.

Solution:

Since this is the same as the probability that the minimum of three independent exponentials, χ_A, χ_B and χ_C, having rates 1/7, 1/2 and 1, respectively, is equal to ξ_C, then

$$P(\text{Next update is from Cindy}) = P(\chi_C = \min\{\chi_A, \chi_B, \chi_C) = \frac{1}{1/7 + 1/2 + 1} = \frac{14}{23}$$

(c) [8 PTS] Given that yesterday Joyce received 5 Facebook updates from her group of three friends (during a 24 hour period), compute the probability that none of them are from Cindy.

Solution:

Let $N_A(t)$, $N_B(t)$, and $N_C(t)$ denote the number of Facebook updates from Amy, Brenda and Cindy, respectively, that Joyce received during the period $[0, t]$. Then, we need to compute

$$\begin{aligned}
P(N_C(t) = 0 | N(t) = 5) &= \frac{P(N_C(t) = 0,\ N(t) = 5)}{P(N(t) = 5)} \\
&= \frac{P(N_C(t) = 0, N_A(t) + N_B(t) + N_C(t) = 5)}{P(N(t) = 5)} \\
&= \frac{P(N_C(t) = 0) P(N_A(t) + N_B(t) = 5)}{P(N(t) = 5)} \qquad \text{(by independence)} \\
&= \frac{e^{-1} \cdot e^{-(1/7+1/2)} (1/7 + 1/2)^5 / 5!}{e^{-(23/14)} (23/14)^5 / 5!} \\
&= \frac{(1/7 + 1/2)^5}{(23/14)^5} = \left(\frac{9}{23}\right)^5
\end{aligned}$$