First Midterm Exam

Place all answers on the question sheet provided. The exam is open textbook and open notes/handouts/ homework. You are allowed to use a calculator, but not a computer, tablet or smartphone. Write all answers clearly and in complete sentences. All answers should be supported by analysis or an argument. This exam has a total of 53 points.

1. Alan owns a store in the neighborhood and he believes that there are good days, when many customers come into his store, and bad days, when very few do. He models this customer behavior using a Markov chain, where he has estimated that a good day will be followed by another good day with probability 0.6, and a bad day will be followed by another bad day with probability 0.5, independently of previous days. Let $X_n = 1$ if the nth day is a good day, and $X_n = 2$ if it is a bad day.

   (a) [6 PTS] Write the one-step transition probability matrix.
   Solution:

$$P = \begin{pmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{pmatrix}.$$

(b) [7 PTS] Suppose that the 2-step transition matrix is

$$P^2 = \begin{pmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{pmatrix}.$$

Today (Thursday) was a really bad day, since no customers showed up at all. Alan believes that it was related to the very bad weather they experienced, and since this bad weather is likely to persist until tomorrow, he believes that tomorrow will be a bad day again with probability 0.9, i.e., $P(X_0 = 1) = 0.1$ and $P(X_0 = 2) = 0.9$. Compute the probability that this coming Sunday is a good day.

Solution:

The question is asking for the state of the Markov chain on Sunday, when X_0 is a Friday, therefore, the question can be written as $P(X_2 = 1)$. To compute the distribution of X_2 we need to compute

$$qP^2 = (0.10.9) \begin{pmatrix} 0.56 & 0.44 \\ 0.55 & 0.45 \end{pmatrix} = (0.551, 0.449).$$

The answer we want is $P(X_2 = 1) = 0.551$.

(c) [6 PTS] Alan's store has an average revenue of \$1000 on a good day, and \$300 on a bad day. If the stationary distribution of the chain is $(\pi_1, \pi_2) = (5/9, 4/9)$, compute the long-run average revenue of Alan's store.

Solution:

The long-run average profit is given by

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i),$$

where $f(1) = 1000$ and $f(2) = 300$. Therefore, the answer we seek is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \pi_1 f(1) + \pi_2 f(2) = \frac{5}{9}(1000) + \frac{4}{9}(300) = \frac{6200}{9} = 688.8889$$

2. Recall the urns and balls problem from Lecture 4, where we have two urns, A and B, and N balls. On each round of the game we pick randomly one of the N balls and move it to the "other" urn. Let X_n be the number of balls in urn A after n rounds of the game. The transition probabilities for $\{X_n : n \geq 1\}$ are:

$$p(i, i+1) = \frac{N-i}{N}, \qquad 0 \leq i \leq N-1,$$
$$p(i, i-1) = \frac{i}{N}, \qquad 1 \leq i \leq N.$$

For each of the following justify your answer:

(a) [5 PTS] Is this Markov chain irreducible?
Solution:
Yes, between any states $i < j$ the path that goes

$$i \to i+1 \to i+2 \to \cdots \to j$$

has positive probability:

$$\frac{N-i}{N} \cdot \frac{N-(i+1)}{N} \cdot \frac{N-(i+2)}{N} \cdots \frac{N-(j-1)}{N} > 0$$

Similarly, the reversed path has probability

$$\frac{j}{N} \cdot \frac{j-1}{N} \cdot \frac{j-2}{N} \cdots \frac{i+1}{N} > 0.$$

(b) [5 PTS] Is this Markov chain aperiodic?
Solution:
No, since $p(i,i)^{2n+1} = 0$ for any $n \geq 1$, while $p(i,i)^{2n} > 0$.

(c) [5 PTS] Is this Markov chain time-reversible?

Solution:

Yes, it is a birth-and-death chain on a finite state space, therefore, from the lectures, it is time-reversible.

3. One of your friends is having a birthday party. The number of guests, besides yourself, is a Poisson random variable with mean 10. You are trying to decide whether or not bring a gift for your friend, and you estimate that each of the guests will bring a gift with probability 0.3, independently of each other.

   (a) [8 PTS] Compute the mean and variance of the number of gifts your friend will get, not counting your own.

   Solution:

   Let N be the number of guests invited to the party, other than yourself. N is a Poisson random variable with mean 10. Now let G be the number of gifts, without counting yours, that your friend receives. Then, conditionally on $N = n$, we have that G is a Binomial$(n, 0.3)$. It follows that

   $$g(n) = E[G|N = n] = 0.3n \qquad \text{and} \qquad h(n) = \text{var}(G|N = n) = (0.3)(0.7)n.$$

   It follows from the iterated expectation formula that

   $$E[G] = E[E[G|N]] = E[g(N)] = E[0.3N] = 0.3E[N] = 0.3(10) = 3$$

   and from the total variance formula that

   $$\begin{aligned}\text{var}(G) &= E[\text{var}(G|N)] + \text{var}(E[G|N]) = E[h(N)] + \text{var}(g(N)) = E[0.21N] + \text{var}(0.3N)\\ &= 0.21E[N] + (0.3)^2\text{var}(N) = 0.21(10) + 0.09(10) = 2.1 + 0.9 = 3\end{aligned}$$

   Alternatively, we showed in class that if N is a Poisson random variable with mean λ, and $G|N = n$ is Binomial(n, p), then G is a Poisson random variable with mean λp. Hence,

   $$E[G] = 10(0.3) = 3 \qquad \text{and} \qquad \text{var}(G) = 10(0.3) = 3.$$

(b) [8 PTS] If you decide not to get a gift for your friend, what is the probability that you are the only guest not bringing a gift (which would be embarrassing)?

Solution:

By conditioning on the number of guests other than yourself we have

$$
\begin{aligned}
&P(\text{you are the only guest not bringing a gift})\\
&= \sum_{n=0}^{\infty} P(\text{you are the only guest not bringing a gift}|N = n)P(N = n)\\
&= \sum_{n=0}^{\infty} P(G = n|N = n)P(N = n)\\
&= \sum_{n=0}^{\infty} (0.3)^n P(N = n)\\
&= \sum_{n=0}^{\infty} (0.3)^n \frac{e^{-10}(10)^n}{n!}\\
&= e^{-10} \sum_{n=0}^{\infty} \frac{3^n}{n!} = e^{-10}e^3 = e^{-7}.
\end{aligned}
$$