Midterm #1, Physics 5C, Spring 2018. Write your responses below, on the back, or on the extra pages. Show your work, and take care to explain what you are doing; partial credit will be given for incomplete answers that demonstrate some conceptual understanding. Cross out or erase parts of the problem you wish the grader to ignore.

Problem 1: (20 pts)

A particle is in an initial state with a wavefunction[1]

$$\Psi(x,t=0) \quad = \quad A\sqrt{x}e^{iax} \qquad \text{for } 0 < x < L \qquad \Psi(x,t=0) = 0 \ \text{ otherwise} \tag{1}$$

where A and a are real constants.

1a) Determine the constant A

Normalization requires

$$\int_{-\infty}^{\infty} \Psi^* \Psi dx = \int_0^L |A|^2 x dx = |A|^2 \frac{L^2}{2} = 1 \tag{2}$$

so

$$A = \frac{\sqrt{2}}{L} \rightarrow \Psi(x,t=0) = \frac{\sqrt{2}}{L}\sqrt{x}e^{iax} \tag{3}$$

up to an arbitrary complex phase, which we choose to be zero.

1b) Determine the probability that a measurement finds the particle between $x = 0$ and $x = L/2$.

The probability is

$$\int_0^{L/2} \Psi^* \Psi dx = \int_0^{L/2} \frac{2}{L^2} x dx = \left. \frac{x^2}{L^2} \right|_0^{L/2} = \frac{1}{4} \tag{4}$$

1c) An experimenter sets up a very large number of particles, each with the wavefunction $\Psi(x,t=0)$, and measures the position x of each. What is the average value of these measurements?

The problem is asking for the expectation value

$$\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* \Psi x dx = \int_0^L \frac{2}{L^2} x^2 dx = \left. \frac{2}{3}\frac{x^3}{L^2} \right|_0^L = \frac{2}{3}L \tag{5}$$

1d) Find the minimum possible uncertainty (i.e. standard deviation) σ_p of the momentum of the particle.

The uncertainty principle gives $\sigma_p \geq \hbar/2\sigma_x$. So we calculate the uncertain in position

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} \Psi^* \Psi x^2 dx = \int_0^L \frac{2}{L^2} x^3 dx = \left. \frac{2}{4}\frac{x^4}{L^2} \right|_0^L = \frac{1}{2}L^2 \tag{6}$$

The uncertainty is then

$$\sigma_x^2 = \left\langle x^2 \right\rangle - \langle x \rangle^2 = \frac{L^2}{2} - \frac{4L^2}{9} = \frac{9L^2}{18} - \frac{8L^2}{18} = \frac{L^2}{18} \tag{7}$$

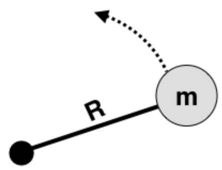
so $\sigma_x = L/\sqrt{18}$ Then by the uncertainty principle $\sigma_p \geq \hbar/2\sigma_x$ and the minimum possible value is

$$\sigma_p = \sqrt{18}\hbar/2L \tag{8}$$

[1] Strictly speaking, this is not a good wavefunction since it has discontinuities, e.g. at $x = L$. But this issue does not affect any part of this problem, so don't worry about it. We can consider the given $\psi(x)$ as an approximation to a wavefunction that smoothly but sharply drops to zero at $x = L$.

Problem 2: (6 pts)

A particle of mass m attached to a (massless) rigid rod of length R. The particle can rotate in a circle around the center at the fixed distance R. The classical expression for the particle energy is $E = (1/2)mv^2$ and for angular momentum $L = mvR$.



2a) Use Bohr's approach of combining angular momentum quantization with the classical expressions for E and L to determine the possible rotational energy states of this particle.

Bohr's quantized angular momentum in units of hbar, $L = n\hbar$. From the classical expression $L = mvR$ this implies $mvR = n\hbar$, or $v = n\hbar/mR$. The energy is then

$$E = \frac{1}{2}mv^2 = \frac{1}{2}\frac{mn^2\hbar^2}{m^2R^2} = \boxed{\frac{n^2\hbar^2}{2mR^2}} \tag{9}$$

This expression is very similar to the particle in the box energies (just without a factor of π^2) so we know the units are correct.

2b) The particle can make a transition (a "quantum leap") from a higher rotational energy state to a lower energy state and emit light (a photon). What is the spectrum – i.e., the possible values of the angular frequencies, ω, of photons that can be emitted[2]?

The frequency of the photon is $\hbar\omega = \Delta E$ where ΔE is the difference in energy between the two states. From the above we have

$$\Delta E = E_a - E_b = \frac{n_a^2\hbar^2}{2mR^2} - \frac{n_b^2\hbar^2}{2mR^2} \tag{10}$$

where n_a and n_b are integers with $n_a > n_b$ (to make sure the change in energy is positive). So the frequency of the photon is

$$\omega = \frac{\hbar}{2mR^2}(n_a^2 - n_b^2) \tag{11}$$

[2]This simple system is not so bad an approximation for diatomic molecules – like carbon and oxygen bonding to form CO (carbon-monoxide) – which physicists probe by looking at the spectral lines from quantized rotational energy states.

Problem 3: (18 pts)

Schrodinger's initial attempt at writing down an equation (before he came up with the Schrodinger equation) was an equation of the sort

$$\hbar^2 \frac{\partial^2 \Psi}{\partial t^2} = \hbar^2 c^2 \frac{\partial^2 \Psi}{\partial x^2} - m^2 c^4 \Psi - V(x)\Psi \tag{12}$$

where m is the mass of the particle and c the speed of light.

3a) What is the dispersion relation between the ω and k of monochromatic waves for this equation when $V(x) = 0$?

We plug in a monochromatic wave solution $\Psi = Ae^{ikx-\omega t}$. Note that each time derivative gives a factor $-i\omega$ and each space derivative a factor ik. We get

$$\hbar^2(-i\omega)^2\Psi = \hbar^2 c^2 (ik)^2 \Psi - m^2 c^4 \Psi \tag{13}$$

or

$$-\hbar^2\omega^2 = -\hbar^2 c^2 k^2 - m^2 c^4 \tag{14}$$

$$\hbar^2\omega^2 = \hbar^2 c^2 k^2 + m^2 c^4 \tag{15}$$

So the dispersion relation is

$$\omega(k) = \sqrt{c^2k^2 + m^2c^4/\hbar^2} \tag{16}$$

Notice that using the Einstein-Debroglie relations $E = \hbar\omega$, $p = \hbar k$ the dispersion relation can be written

$$E^2 = p^2c^2 + m^2c^4 \tag{17}$$

which is the relationship between energy and momentum in special relativity. So this equation (called the Klein-Gordan equation) is an attempt at a special relativistic equation – unfortunately it does not describe electrons but spin 0 particles.

3b) A wavepacket is constructed by summing up monochromatic waves narrowly spread around a wavenumber k_0 (i.e., with wavenumbers between $k_0 - \Delta_k$ and $k_0 + \Delta_k$). How fast does this wave packet move?

The wave packet velocity is given by the group velocity

$$v_g = \frac{\partial \omega}{\partial k} = \frac{c^2 k_0}{\sqrt{c^2k_0^2 + m^2c^4/\hbar^2}} \tag{18}$$

Or writing in terms of Debroglie momentum $p = hk_0$

$$v_g = \frac{c^2 p}{\sqrt{c^2p^2 + m^2c^4}} = \frac{c}{\sqrt{1 + m^2c^4/p^2}} \tag{19}$$

We see that as $p \to \infty$ that $v_g \to c$. So this group velocity can never exceed c, in accordance with special relativity (and in contrast to the Schrodinger equation group velocity $v_g = \hbar k_0/m = p/m$).

Using separation of variables, we can write the time-independent form of this equation in terms of $\psi(x)$, the spatial part of the wavefunction

$$-\hbar^2 c^2 \frac{\partial^2 \psi}{\partial x^2} + m^2c^4\psi(x) + V\psi(x) = E^2\psi(x) \tag{20}$$

where E is the energy. Consider the infinite square well potential (aka particle in a box) where $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ otherwise.

3c) Assuming $E^2 > m^2c^4$, solve this time-independent equation to determine the possible energies[3] of a particle in the infinite square well.

[3] You may notice that, mathematically, the solutions allow for negative energies, which is one of the issues that made Schrodinger move on.

For $V = 0$ we write the equation inside the well as

$$-\hbar^2 c^2 \frac{\partial^2 \psi}{\partial x^2} = (E^2 - m^2c^4)\psi(x) \tag{21}$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{c^2\hbar^2}(E^2 - m^2c^4)\psi(x) = -k^2\psi(x) \tag{22}$$

where

$$k = \frac{\sqrt{(E^2 - m^2c^4)}}{\hbar c} \tag{23}$$

Solution is

$$\psi(x) = A\sin(kx) + B\sin(kx) \tag{24}$$

Applying the boundary conditions, we have $\psi(0) = 0$ implies $B = 0$ so $\psi(x) = A\sin(kx)$ and the BC $\psi(L) = 0$ gives

$$k = \frac{n\pi}{L} \tag{25}$$

This implies

$$\frac{n^2\pi^2}{L^2} = \frac{(E^2 - m^2c^4)}{\hbar^2c^2} \tag{26}$$

$$\frac{n^2\hbar^2c^2\pi^2}{L^2} = (E^2 - m^2c^4) \tag{27}$$

and so

$$E = \pm\sqrt{\frac{n^2\hbar^2c^2\pi^2}{L^2} + m^2c^4} \tag{28}$$

3d) Show that states with $E^2 < m^2c^4$ cannot exist.

If $E^2 < m^2c^4$ we have the equation

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{1}{c^2\hbar^2}(E^2 - m^2c^4)\psi(x) = \alpha^2\psi(x) \tag{29}$$

where

$$\alpha = \frac{\sqrt{(m^2c^4 - E^2)}}{\hbar c} \tag{30}$$

is a real number (alternatively we see that the quantity k defined in part 3c becomes imaginary, so we could define $\alpha = i\kappa$) The solutions are now the real exponentials

$$\psi(x) = Ae^{-\alpha x} + Be^{\alpha x} \tag{31}$$

We can see that as $x \to \infty$ the second term blows up instead of going to zero (as required by the BCs of the infinite square well), so this implies $B = 0$. But as $x \to -\infty$ the first term blows up, implying $A = 0$. So there is no solution that meets the boundary conditions except for the trivial case $\psi(x) = 0$ (which doesn't represent any particle at all).

One can alternatively apply the infinite square well boundary conditions specifically to get the same result. At $x = 0$ we have

$$A + B = 0 \to A = -B \tag{32}$$

and at $x = L$

$$Ae^{-\alpha L} - Ae^{\alpha L} = 0 \to Ae^{-\alpha L}(1 - e^{2\alpha L}) = 0 \tag{33}$$

since $e^{-\alpha L}$ is never zero we can divide both sides to get

$$A(1 - e^{2\alpha L}) = 0 \tag{34}$$

So either $A = 0$ or $e^{2\alpha L} = 1$. But in the latter case taking the natural log of both sides shows that $2\alpha L = 0$ or $\alpha = 0$. From the definition of α this implies $E^2 = m^2c^4$, in conflict with assumption $E^2 < m^2c^4$. So there is no (non-trivial) solution that meets the boundary conditions.

Problem 4: (8 pts)

Consider the potential barrier

$$V(x) = 0 \ \ \text{(for } x < 0) \qquad V(x) = V_0 \ \text{ for } 0 < x < L \qquad V(x) = 0 \ \text{ for } x > L \tag{35}$$

A particle with energy $E > V_0$ is incident on this barrier from the left (i.e., from $x = -\infty$)

4a) Write down the solutions to time-independent Schrodinger Eq. in each of the 3 regions of the potential. Call regions from left to right regions 1,2,3. The solutions are

$$\psi_1(x) = Ae^{ik_1x} + Be^{-ik_1x} \tag{36}$$
$$\psi_2(x) = Ce^{ik_2x} + De^{-ik_2x} \tag{37}$$
$$\psi_3(x) = Fe^{ik_1x} + Ge^{-ik_1x} \tag{38}$$
$$\tag{39}$$

where

$$k_1 = \sqrt{2mE/\hbar^2} \quad k_2 = \sqrt{2m(E - V_0)/\hbar^2} \tag{40}$$

4b) Write down the boundary conditions that relate the coefficients of the solution. You can assume that there is no particle incident from the right (i.e., from $x = \infty$).

We ignore any wave incident from the right, so set $G = 0$. The boundary conditions at $x = 0$ are

$$A + B = C + D \tag{41}$$
$$ik_1(A - B) = ik_2(C - D) \tag{42}$$

and at $x = L$

$$Ce^{ik_2L} + De^{-ik_2L} = Fe^{ik_1L} \tag{43}$$
$$ik_2(Ce^{ik_2L} - De^{-ik_2L}) = ik_1Fe^{ik_1L} \tag{44}$$
$$\tag{45}$$

or written all together more neatly

$$A + B = C + D \tag{46}$$
$$A - B = \frac{k_2}{k_1}(C - D) \tag{47}$$
$$Ce^{ik_2L} + De^{-ik_2L} = Fe^{ik_1L} \tag{48}$$
$$Ce^{ik_2L} - De^{-ik_2L} = \frac{k_1}{k_2}Fe^{ik_1L} \tag{49}$$
$$\tag{50}$$

4c) Assume now that the particle incident from the left has energy $E = \hbar^2\pi^2/2mL^2 + V_0$. Show that for this special energy the particle is *never* reflected from the barrier.

For the energy specified, we see that the wavenumber in region 2 is $k_2 = \pi/L$. Then the $x = L$ boundary conditions become

$$Ce^{i\pi} + De^{-i\pi} = Fe^{ik_1L} \tag{51}$$
$$Ce^{i\pi} - De^{-i\pi} = \frac{k_1}{k_2}Fe^{ik_1L} \tag{52}$$
$$\tag{53}$$

or since $e^{i\pi} = e^{-i\pi} = -1$

$$C + D = -Fe^{ik_1 L} \tag{54}$$

$$C - D = -\frac{k_1}{k_2} Fe^{ik_1 L} \tag{55}$$

$$\tag{56}$$

In the BCs at $x = 0$ we also have terms $(C + D)$ and $(C - D)$ so combining the two sets gives

$$A + B = -Fe^{ik_1 L} \tag{57}$$

$$A - B = -Fe^{ik_1 L} \tag{58}$$

These two equations imply

$$A + B = A - B \rightarrow B = -B \rightarrow B = 0 \tag{59}$$

So the reflection coefficient $R = |B|^2/|A|^2 = 0$

Problem 5:

A particle is in the $n = 2$ state of the symmetric infinite square well where $V(x) = 0$ for $-L/2 < x < L/2$ and $V(x) = \infty$ otherwise. All of the sudden, both sides of the well are moved outward such that the well doubles in size, i.e. $V(x) = 0$ for $-L < x < L$ and $V(x) = \infty$. Assume the wavefunction of the particle remains unchanged during this transition.

5a) What is the probability that a measurement of the particle energy in the expanded box yields an energy E_n^{exp}, where E_n^{exp} are the energy of stationary states (i.e., energy eigenvalues) in the expanded box? You don't have to carry out integrations, but properly set up any integrals you may need.

This problem is very similar to the "parabola in a well" problem that you did on the homework, just with a different initial wavefunction. Here the initial wavefunction of the particle is that of the $n = 1$ state in the small well. For the symmetric well, the energy eigenstates are alternatively even (cosines) and odd (sines). The ground state is even

$$\psi(x) = \sqrt{\frac{2}{L}}\cos(\pi x/L) \;\; \text{for} \;\; -L/2 < x < L/2 \qquad \psi(x) = 0 \;\; \text{otherwise} \tag{60}$$

The energy eigenstates in the expanded box the wavefunctions are also the standard ones for the symmetric infinite square well, just with $L \to 2L$, which gives

$$\psi_n^{\text{exp}}(x) = \sqrt{\frac{1}{L}}\cos(n\pi x/2L) \quad n = 1, 3, 5, ... \;\; \text{(even states)} \tag{61}$$

$$\psi_n^{\text{exp}}(x) = \sqrt{\frac{1}{L}}\sin(n\pi x/2L) \quad n = 2, 4, 6... \;\; \text{(odd states)} \tag{62}$$

The initial wavefunction can be written as a superposition of the energy eigenstates of the expanded well

$$\psi(x) = \sum_n c_n \psi_n^{\text{exp}}(x) \tag{63}$$

To determine the coefficients we use Fourier's trick. For odd states these are

$$c_n = \int \psi(x)\psi_n^{\text{exp}}(x)dx = \int_{-L/2}^{L/2} \sqrt{\frac{2}{L}}\cos(\pi x/L)\sqrt{\frac{1}{L}}\sin(n\pi x/2L)dx \tag{64}$$

$$c_n = \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2} \cos(\pi x/L)\sin(n\pi x/2L)dx \tag{65}$$

The limits are from $-L/2$ to $L/2$ since the initial wavefunction is zero outside of this. For even states, Fourier's trick gives

$$c_n = \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2} \cos(\pi x/L)\cos(n\pi x/2L)dx \tag{66}$$

The lowest energy state is given by $n = 1$ in the expanded well, which has a coefficeint

$$c_1 = \frac{\sqrt{2}}{L}\int_{-L/2}^{L/2} \cos(\pi x/L)\cos(\pi x/2L)dx \tag{67}$$

And the probability of measuring the ground energy is given by $|c_1|^2$. Note that for the even states we have an integral of an odd function (sine) times an odd function (cosine) so the integral has to be $c_n = 0$ by symmetry. We saw the same behavior for the "parabola in a box" homework problem.

5b) Before the well being expanded, the particle had some energy E. What is the probability that a measurement of the particle energy *after* the well is expanded yields this same value of energy? Give a number and explain how you arrived at it.

Initially the particle is in the $n = 1$ state of a well of length L, so has energy

$$E_{\text{init}} = \frac{\hbar^2\pi^2}{2mL^2} \tag{68}$$

In the expanded box of length $2L$ the allowed quantized energies are

$$E_{n'}^{\text{exp}} = \frac{\hbar^2 \pi^2 n'^2}{2m(2L)^2} = \frac{\hbar^2 \pi^2 n'^2}{8mL^2} \tag{69}$$

For the final energy to be measured to be the same as the initial energy the particle would need to be in the $n' = 2$ state of the expanded well. This is an odd state, so the $c_n = 0$ as stated above. The probability is hence zero.

The fact that the integral gives zero can be made clear by plotting up the functions below. The integral of the product of the functions for the left side of the well exactly cancels the integral on the right side.