Physics 8B Midterm 1 Solutions

Lecture 1, Spring 2018

Problem 1.

Part A: In order to find the electric field at the two given points, we will use Gauss' law. The charge distribution in this problem in spherically symmetric, so we will choose our Gaussian surfaces to be spheres sharing a common center with the conducting spheres of the problem. Let's give our first Gaussian sphere a radius of r_1 since we are interested in the electric field at this distance. Gauss' law tells us that

$$\oint \vec{E} \cdot \hat{n} dA = \frac{Q_{enc}}{\epsilon_0}.$$

The left-hand side is the electric flux through our chosen surface. Whatever the electric field is at a distance r_1, it must be pointing radially outward. This means the electric field lines are perpendicular to the surface (parallel to $\hat{n}$), and the electric flux integral includes the full surface area of our Gaussian sphere. As for Q_{enc}, our surface only encloses the charged sphere of radius a. Thus we have

$$|\vec{E}_1| 4\pi r_1^2 = \frac{Q_a}{\epsilon_0}$$
$$\Rightarrow |\vec{E}_1| = \frac{1}{4\pi\epsilon_0}\frac{Q_a}{r_1^2}.$$

Since r_1 is below the positively charged sphere, the electric field must point downward:

$$\vec{E}_1 = -\frac{1}{4\pi\epsilon_0}\frac{Q_a}{r_1^2}\hat{y}.$$

As before, we take our Gaussian surface to be a sphere of radius r_2. Most everything is the same, except our surface now encloses both charged spheres and we have

$$|\vec{E}_2| 4\pi r_2^2 = \frac{Q_a + Q_b}{\epsilon_0}$$
$$\Rightarrow |\vec{E}_2| = \frac{1}{4\pi\epsilon_0}\frac{Q_a + Q_b}{r_2^2}.$$

Since r_2 is to the right of the positively charged spheres, the electric field must point to the right:

$$\vec{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{Q_a + Q_b}{r_2^2}\hat{x}.$$

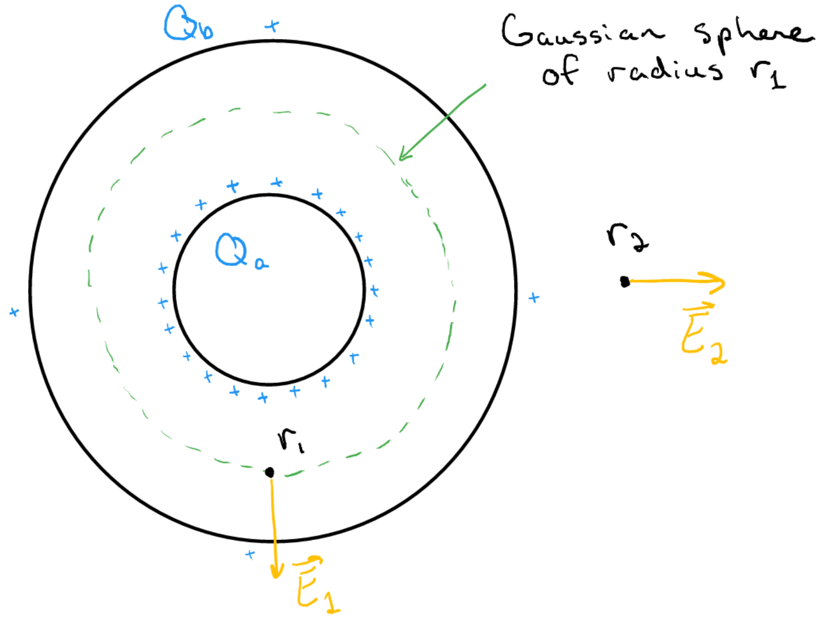

Figure 1: The electric fields for Part A, with the Gaussian surface used to find $\vec{E}_1$.

Part B: When the wire connects both spheres, the whole system is a single conductor. Charges will continue moving until no electric field is present inside the conductor, thus both spheres and the wire are at equal electrical potential. This can only be the case if no net charge remains on the inner sphere of radius a. (Suppose there was net charge on the inner sphere; then an electric field would exist between the two sphere by Gauss' law, and the potential would be different between the spheres.) Using this fact and conservation of charge,

$$
\begin{aligned}
Q'_a &= 0 \\
Q'_b &= Q_a + Q_b.
\end{aligned}
$$

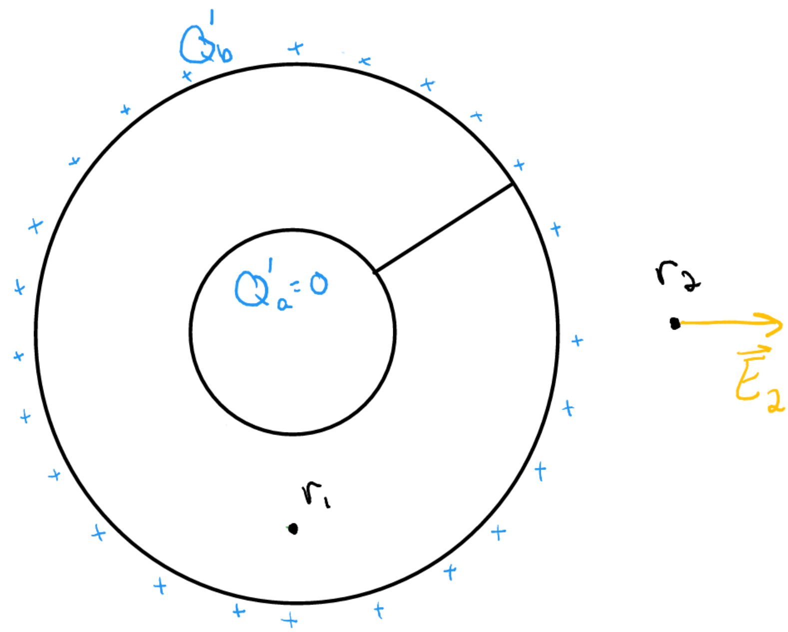

Figure 2: Now that a wire connects the spheres, all excess charge moves to the outer sphere. $\vec{E}_1$ is now zero, and $\vec{E}_2$ remains unchanged.

Part C: We find the electric fields exactly as we did in (a), but $Q_a \to Q_a'$ and $Q_b \to Q_b'$. Thus at r_1,

$$\begin{aligned}|\vec{E}_1| &= \frac{1}{4\pi\epsilon_0}\frac{Q_a'}{r_1^2} \\ &= \frac{1}{4\pi\epsilon_0}\frac{0}{r_1^2} \\ &= 0\end{aligned}$$

and at r_2,

$$\begin{aligned}\vec{E}_2 &= \frac{1}{4\pi\epsilon_0}\frac{Q_b'}{r_2^2}\hat{x} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q_a + Q_b}{r_2^2}\hat{x}.\end{aligned}$$

Problem 2.

Part A: Note that we can treat the given charge configuration as the sum of a larger sphere (radius $2a$) with charge density ρ and a smaller sphere (radius a) with charge density -8ρ. At point A, we can find the electric field produced by each of these spheres individually and then add them to find the net electric field. Since we are outside these uniformly charged spheres, we may treat them as point charges located at the center of each sphere.

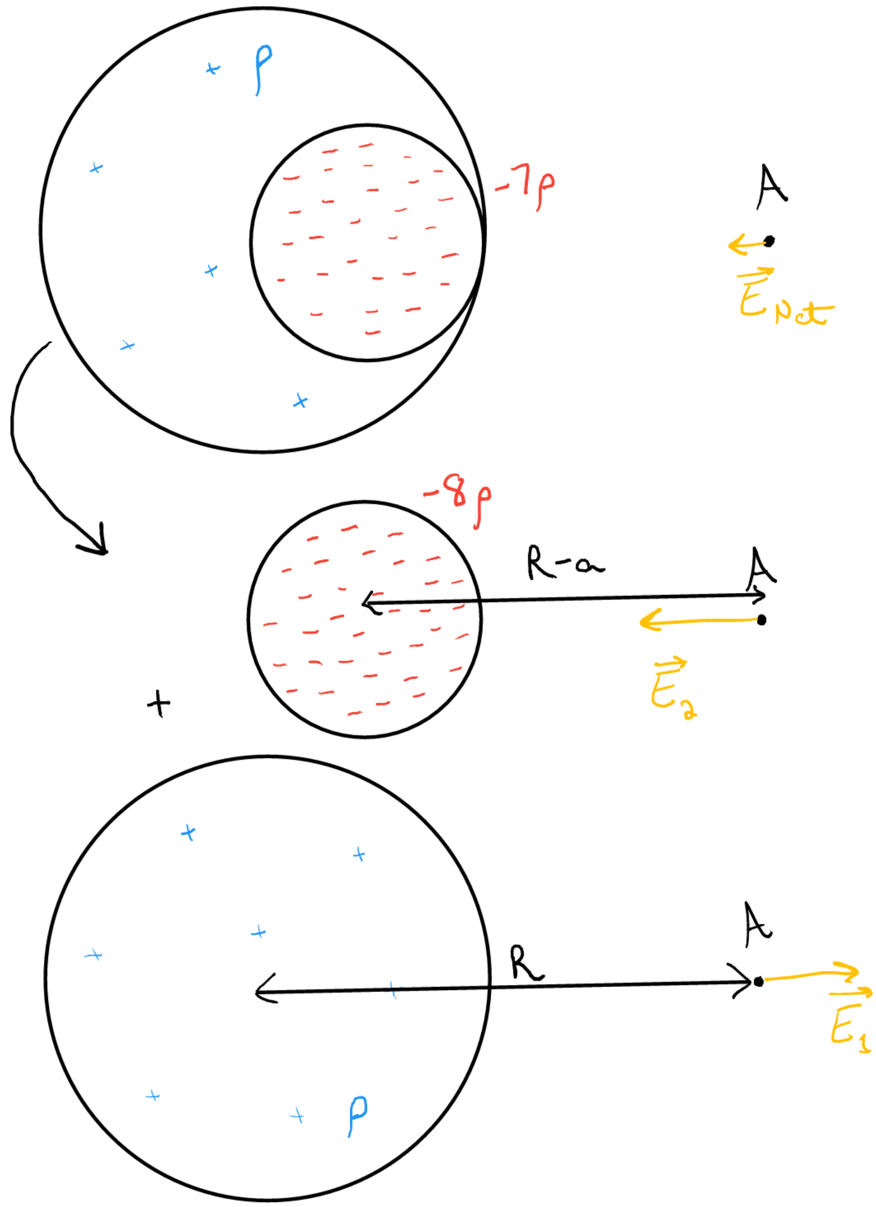

Figure 3: Separation of the original charge configuration into two uniformly charged spheres.

Let's call the charge on the larger sphere Q_1; it is found by multiplying the charge density with the volume of the sphere:

$$\begin{aligned} Q_1 &= \rho\Big(\frac{4}{3}\pi(2a)^3\Big) \\ &= 8\rho\Big(\frac{4}{3}\pi a^3\Big), \end{aligned}$$

and let's call the charge on the smaller sphere Q_2:

$$Q_2 = -8\rho\Big(\frac{4}{3}\pi a^3\Big).$$

Notice that Q_1 and Q_2 are equal in magnitude, so let's define $Q \equiv |Q_1| = |Q_2|$. At point A, we are a distance R away from the center of the larger sphere:

$$\begin{aligned} \vec{E}_1 &= \frac{1}{4\pi\epsilon_0}\frac{Q_1}{R^2}\hat{x} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{x}, \end{aligned}$$

and a distance $R-a$ away from the center of the smaller sphere:

$$\begin{aligned} \vec{E}_2 &= \frac{1}{4\pi\epsilon_0}\frac{Q_2}{(R-a)^2}\hat{x} \\ &= -\frac{1}{4\pi\epsilon_0}\frac{Q}{(R-a)^2}\hat{x}. \end{aligned}$$

We will need to use the approximation given in the problem to simplify $\vec{E}_2$:

$$\begin{aligned} \frac{1}{(R-a)^2} &= \frac{1}{R^2(1-\frac{a}{R})^2} \\ &\approx \frac{1}{R^2}[1+2\frac{a}{R}], \end{aligned}$$

where we take $u = -\frac{a}{R}$ and $p = -2$ in the approximation $(1+u)^p \approx 1 + pu$. Note this approximation is valid since $R \gg a$ implies $u \ll 1$. The net electric field at A is then

$$\begin{aligned} \vec{E}_{net} &= \vec{E}_1 + \vec{E}_2 \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{x} - \frac{1}{4\pi\epsilon_0}\frac{Q}{(R-a)^2}\hat{x} \\ &= \frac{Q}{4\pi\epsilon_0}\Big(\frac{1}{R^2} - \frac{1}{(R-a)^2}\Big)\hat{x} \\ &\approx \frac{Q}{4\pi\epsilon_0}\Big(\frac{1}{R^2} - \frac{1}{R^2}[1+2\frac{a}{R}]\Big)\hat{x} \\ &= -\frac{Q}{4\pi\epsilon_0}\frac{2a}{R^3}\hat{x}. \end{aligned}$$

Part B: Along the axis of a dipole, the electric field at a distance R is given by

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0}\frac{2\vec{p}}{R^3},$$

where $\vec{p}$ is the dipole moment. Comparing this to our result for $\vec{E}_{net}$, we see that

$$\vec{p} = -Qa\hat{x}.$$

We could also recall that the dipole moment of a $+Q$ charge at a distance a from a $-Q$ charge is defined to be $p = Qa$. The direction of $\vec{p}$ points from the negative charge to the positive charge by convention.

Problem 3.

With the switch initially closed, the voltage supply has had time to fully charge the capacitor. Let us call the capacitance of the capacitor C_0 without the dielectric. Kirchoff's law tells us that the potential difference across the capacitor is the same as the voltage supply, V_b. Thus the energy stored on the capacitor is

$$U_0 = \frac{1}{2}C_0V_b^2.$$

When we insert the dielectric, the capacitance increases by a factor of κ. The new capacitance is

$$C' = \kappa C_0.$$

At this time the charge stored on the capacitor is now

$$Q' = C'V_b.$$

At this point, the capacitor has been fully charged by the voltage supply again (the higher capacitance C' allows for additional charge to flow onto the capacitor). Once we open the switch, the circuit is broken and this charge Q' cannot leave the capacitor. We now remove the dielectric, and the capacitance changes back to C_0. Without the voltage supply attached, the potential difference across the capacitor isn't necessarily V_b. In calculating the final stored energy, we will instead use $V = \frac{Q}{C}$ to find the potential difference:

$$\begin{aligned} U_f &= \frac{1}{2}C_0V^2 \\ &= \frac{1}{2}C_0\left(\frac{Q'}{C_0}\right)^2 \\ &= \frac{1}{2}\frac{Q'^2}{C_0} \\ &= \frac{1}{2}\frac{(C'V_b)^2}{C_0}. \end{aligned}$$

We now have our desired result

$$\begin{aligned}\frac{U_f - U_0}{U_0} &= \frac{\frac{1}{2}\frac{(C'V_b)^2}{C_0} - \frac{1}{2}C_0V_b^2}{\frac{1}{2}C_0V_b^2} \\ &= \frac{\frac{(C')^2}{C_0} - C_0}{C_0} \\ &= \left(\frac{C'}{C_0}\right)^2 - 1 \\ &= \kappa^2 - 1 \\ &= 24.\end{aligned}$$

Problem 4.
As in problem 1, we will use Gauss' law to find the electric field at various distances from the center-axis of the infinite cylinders. To match the symmetry of this charge distribution, our Gaussian surfaces will be cylinders of length L and radius r, sharing a common center-axis with the infinite cylinders. (Don't worry about L, this variable we introduce will go away before the final answer.) Gauss' law tells us that

$$\oint \vec{E} \cdot \hat{n} dA = \frac{Q_{enc}}{\epsilon_0}.$$

The left-hand side is the electric flux through our chosen surface. Whatever the electric field is at a distance r, it must be pointing radially outward (if it's non-zero). This means the electric field lines are perpendicular to the curved side of our Gaussian cylinder (parallel to $\hat{n}$), but parallel to the top and bottom of the Gaussian cylinder (perpendicular to $\hat{n}$). Therefore our electric flux integral only includes the area of the curved side:

$$\oint \vec{E} \cdot \hat{n} dA = |\vec{E}|(2\pi r L).$$

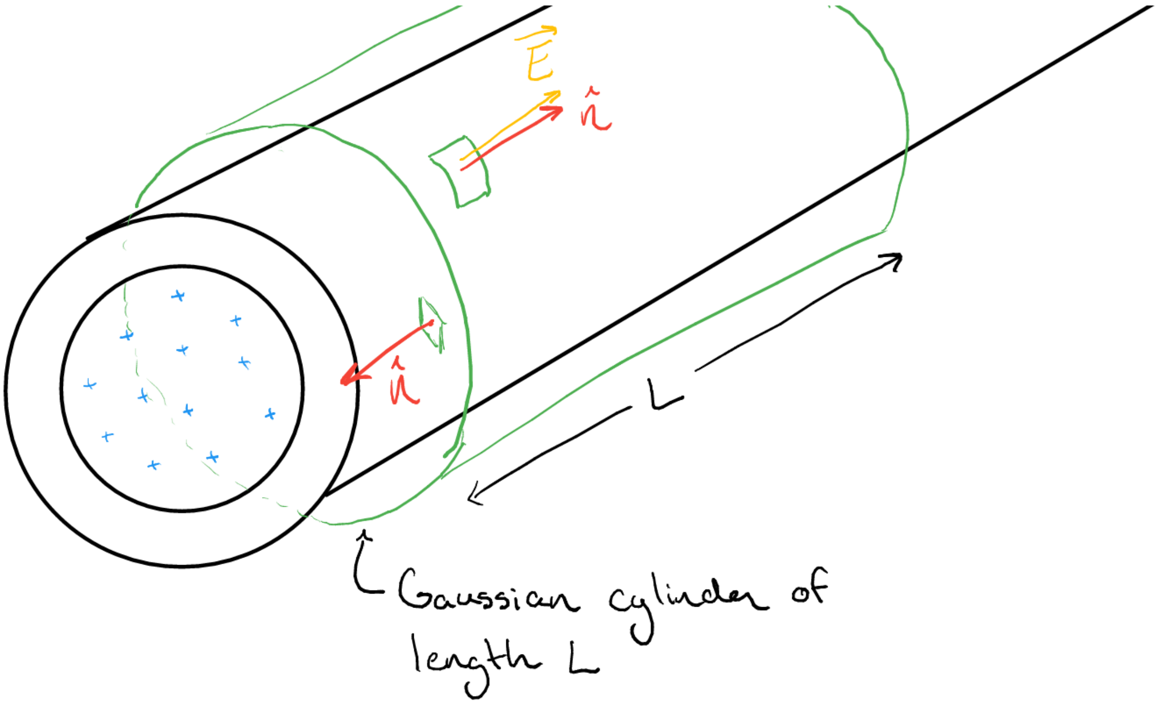

Figure 4: Electric fields pointing radially outward will only 'pierce' through the curved side of the Gaussian cylinder. (Surface charge distributions have not been marked on the conducting shell.)

With this, we just need to see how Q_{enc} changes for various values of r. When $r > b$, our surface encloses a slice of the infinite cylinder with length L. We get the enclosed charge by multiplying charge density by the enclosed volume:

$$Q_{enc} = \rho(\pi a^2 L).$$

Gauss' law then gives us

$$|\vec{E}|(2\pi r L) = \frac{\rho(\pi a^2 L)}{\epsilon_0}$$
$$\Rightarrow |\vec{E}| = \frac{1}{2\epsilon_0}\frac{a^2\rho}{r}.$$

When $a < r < b$, we are inside a conductor and $|\vec{E}| = 0$. When $r < a$, our surface no longer encloses a full slice of the infinite cylinder with length L. The enclosed volume now depends on r: $V_{enc} = \pi r^2 L$. Again, multiply by charge density, we get

$$Q_{enc} = \rho(\pi r^2 L).$$

Gauss' law then gives us

$$|\vec{E}|(2\pi r L) = \frac{\rho(\pi r^2 L)}{\epsilon_0}$$
$$\Rightarrow |\vec{E}| = \frac{1}{2\epsilon_0}\rho r.$$

Adding in the radial direction of our electric field, our final expression is

$$\vec{E}(\vec{r}) = \begin{cases} \frac{1}{2\epsilon_0}\rho r\hat{r} & r < a \\ \vec{0} & a < r < b \\ \frac{1}{2\epsilon_0}\frac{a^2\rho}{r}\hat{r} & r > b \end{cases}$$

Let's choose the zero of potential to be at the center (V(0) = 0). Now, let's find the electric potential as a function of r, the distance from the center of the cylinder. We are finding the potential difference in moving from $r = 0$ to some value of r:

$$V(r) - V(0) = \Delta V = -\int_0^r \vec{E}\cdot d\vec{l}$$
$$\Rightarrow V(r) = -\int_0^r \vec{E}\cdot d\vec{l}.$$

Our electric field changes depending on which region we are in, so we will need to split up this integral into the different regions. When $r < a$,

$$V(r) = -\int_0^r \frac{1}{2\epsilon_0}\rho r dr$$
$$= -\frac{\rho}{2\epsilon_0}\frac{r^2}{2}\Big|_0^r$$
$$= -\frac{\rho}{4\epsilon_0}r^2.$$

When $a < r < b$,

$$V(r) = -\int_0^a \vec{E}\cdot d\vec{l} - \int_a^r \vec{E}\cdot d\vec{l}$$
$$= -\int_0^a \frac{1}{2\epsilon_0}\rho r dr - 0$$
$$= -\frac{\rho}{4\epsilon_0}a^2.$$

Notice that the electric potential stays the same inside the conductor, and it's value is just the value at $r = a$. With $r > b$,

$$
\begin{aligned}
V(r) &= -\int_0^a \vec{E}\cdot\vec{dl} - \int_a^b \vec{E}\cdot\vec{dl} - \int_b^r \vec{E}\cdot\vec{dl} \\
&= -\int_0^a \frac{1}{2\epsilon_0}\rho r dr - 0 - \int_b^r \frac{1}{2\epsilon_0}\frac{a^2\rho}{r}dr \\
&= -\frac{\rho}{4\epsilon_0}a^2 - \frac{a^2\rho}{2\epsilon_0}\int_b^r \frac{1}{r}dr \\
&= -\frac{\rho}{4\epsilon_0}a^2 - \frac{a^2\rho}{2\epsilon_0}\ln r\Big|_b^r \\
&= -\frac{\rho}{4\epsilon_0}a^2 - \frac{a^2\rho}{2\epsilon_0}\ln\left(\frac{r}{b}\right).
\end{aligned}
$$

Altogether,

$$
V(r) = \begin{cases} -\frac{\rho}{4\epsilon_0}r^2 & r \le a \\ -\frac{\rho}{4\epsilon_0}a^2 & a \le r \le b \\ -\frac{\rho}{4\epsilon_0}a^2 - \frac{a^2\rho}{2\epsilon_0}\ln\left(\frac{r}{b}\right)\hat{r} & r \ge b \end{cases}
$$

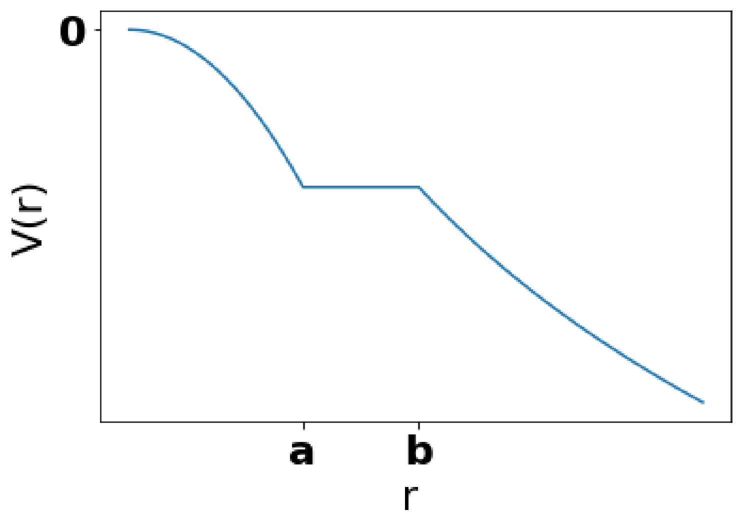

Figure 5: Voltage as a function of r, the distance from the center. The chosen zero point is marked as $V(0) = 0$, but this doesn't really have any physical significance. It's differences in electric potential that physically matter, so the shape of this graph is the important part.

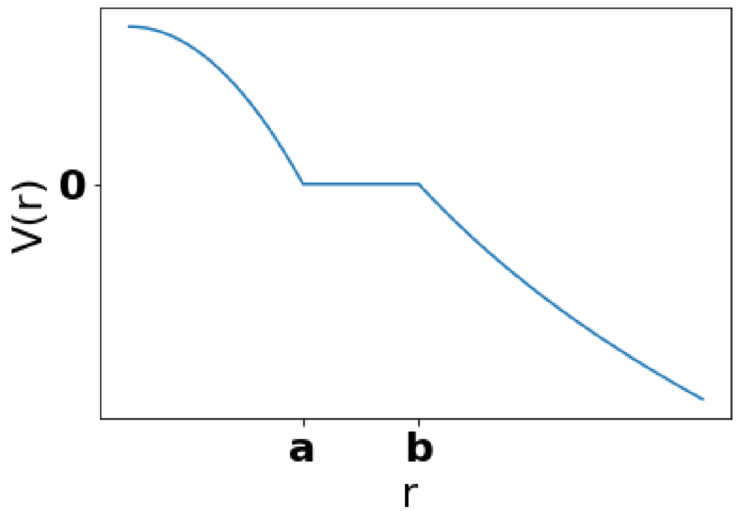

Figure 6: Changing the zero point of potential merely shifts our voltage graph vertically. Another 'nice' choice would be $V = 0$ inside the conductor, which is shown here.