EECS 126 Probability and Random Processes University of California, Berkeley: Spring 2017
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Midterm Exam 2

Last name	First name	SID

Name of student on your left:
Name of student on your right:

- DO NOT open the exam until instructed to do so.
- The total number of points is **110**, but a score of $\geq$ **100** is considered perfect.
- You have 10 minutes to read this exam without writing anything and 105 minutes to work on the problems.
- Box your final answers.
- Partial credit will not be given to answers that have no proper reasoning.
- **Remember to write your name and SID on the top left corner of every sheet of paper.**
- **Do not write on the reverse sides of the pages.**
- All eletronic devices must be turned off. Textbooks, computers, calculators, etc. are prohibited.
- No form of collaboration between students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.
- **You must include explanations to receive credit.**

Problem	Part	Max	Points	Problem	Part	Max	Points
1	(a)	20		2		15	
	(b)	12		3		20	
	(c)	8		4		25	
	(d)	10					
Total						110	

Problem 1. (a) (20 points, 4 points each part) Evaluate the statements with *True* or *False*. Give brief explanations in the provided boxes. Anything written outside the boxes will not be graded.

(1) Given a random variable $M \sim \text{Geom}(\frac{1}{10})$ and i.i.d. exponential random variables $X_i \sim \text{Exp}(1)$, the distribution of the sum $X_1 + X_2 + \cdots + X_M$ is Erlang of order 10 with rate 1.

True or False:
Explanation:

(2) Recall fountain codes as introduced in Lab 4. You would like to recover the packets x_1, x_2, x_3, x_4 and receive the following equations:

$$
\begin{aligned}
&x_2 \oplus x_4 = 1 \\
&x_1 = 1 \\
&x_1 \oplus x_2 \oplus x_3 \oplus x_4 = 1 \\
&x_3 \oplus x_4 = 1
\end{aligned}
$$

The peeling decoder is able to recover all the packets that can possibly be recovered.

True or False:
Explanation:

(3) Give an example of a discrete time Markov Chain which is transient and has period 3.

Answer:

(4) Consider a Poisson process with rate 1. Suppose that the 117th arrival comes at time T. The joint distribution of the first 116 arrivals is the same as the joint distribution of $U_1, U_2, \ldots, U_{116}$, where U_i are i.i.d. $U[0,T]$ random variables.

True or False:

Explanation:

(5) For a random variable X with a well-defined $M_X(s)$, we have:

$$\mathrm{Var}(X) = \frac{\mathrm{d}^2}{\mathrm{d}s^2} \ln M_X(s) \bigg|_{s=0}$$

True or False:

Explanation:

(b) (12 points) After attending an EE126 lecture, you went back home and started playing *Twitch Plays Pokemon*. Suddenly, you realized that you may be able to analyze *Twitch Plays Pokemon*.

Figure 1: A snapshot of 'Twitch Plays Pokemon' - 1

(i) (6 points) The player in the top left corner performs a random walk on the 8 checkered squares and the square containing the stairs. At every step the player is equally likely to move to any of the squares in the four cardinal directions (North, West, East, South) if there is a square in that direction. Find the expected number of moves until the player reaches the stairs in Figure 1.

Figure 2: A snapshot of 'Twitch Plays Pokemon' - 2

(ii) (6 points) The player randomly walks in the same way as in the previous part. Find the probability that the player reaches the stairs in the bottom right corner in Figure 2.

(c) (8 points) Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. random variables and $X_i \sim U[-1, 1]$. Does the sequence $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ converge in probability? If so, what does it converge to?

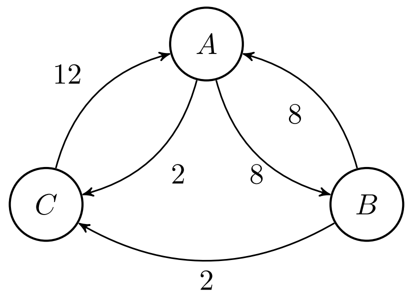
(d) (10 points) Nodes in a wireless sensor network are dropped randomly according to a two-dimensional Poisson process. A two-dimensional Poisson process is defined as follows:

(i.) For any region of area A, the number of nodes in that region has a Poisson distribution with mean λA,

(ii.) The number of nodes in non-overlapping regions is independent.

Suppose that you are at an arbitrary location in the plane. Let X be the distance to the nearest node in the sensor network. Find $E[X]$. (Here distance between the points (x_1, y_1) and (x_2, y_2) is defined as $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$)

Problem 2. (15 points) Consider the continuous time Markov chain given in the figure.



(a.) (7 points) Draw a DTMC (discrete time Markov Chain) which has the same stationary distribution as the given CTMC and find the stationary distribution (it is sufficient to set-up the equations).

(b.) (8 points) Let τ denote the first time the CTMC is in state C after starting from state A at time 0. Find $M_\tau(s)$, the MGF of τ.

Problem 3. (20 points)

(a.) (10 points) Batteries from Company X last an amount of time exponentially distributed with rate λ. The company guarantees that $\lambda \geq 2$. Suppose that you have used n batteries from Company X and observed their lifetimes to be $X_1, X_2, \ldots, X_n$. Using Chebyshev's inequality, find the minimum n required to construct a confidence interval for the mean interval of the light bulb such that it has tolerance at most ϵ and confidence at least $1-\delta$. In other words, we want to find n such that

$$P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - \frac{1}{\lambda}\right| \geq \varepsilon\right) \leq \delta$$

(b.) (10 points) Suppose now that you have taken 10000 samples of the batteries. With the same tolerance level ϵ, use the CLT to find your new confidence (you may again use the company guarantee that $\lambda \geq 2$). You may leave your answer in terms of Φ, the CDF of the standard normal distribution.

Problem 4. (25 points, 5 points each) There are a set of M bins and a set of balls labeled $f_1, f_2, \ldots$. Let $f_{(i)} = \{f_j : j \equiv i \mod M\}$ be the set of balls thrown to bin i. The interarrival time between balls thrown to bin i is exponential with rate λ balls per second, and all interarrival times are independent. The situation is illustrated in Figure 3.

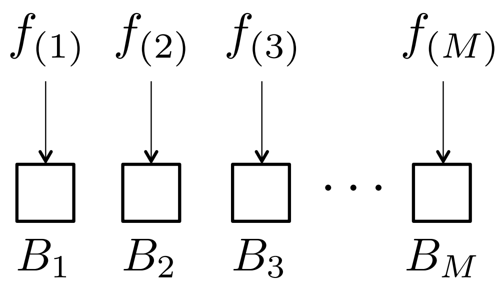


Figure 3: Balls throwing to bins

(a.) Alice and Bob decide to play a betting game. Bob bets that there will be at least one ball in bin B_1 after $\frac{1}{\lambda}$ seconds and Alice bets that there will be at least two balls in bin B_1 after $\frac{2}{\lambda}$ seconds. What is the probability that exactly one of Bob and Alice wins money?

(b.) Suppose that $\lambda = 1$. You peek into bin B_i at a random time t. Let X be the difference between the time of the most recent arrival to bin B_3 and the time of the next arrival to any of the M bins. Find $E[X^2]$.

Your friend finds a switch which can send balls to two consecutive bins as in Figure 4.

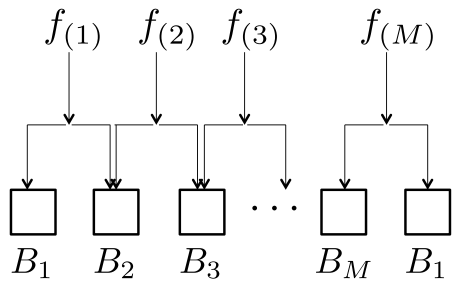


Figure 4: Balls throwing to bins in the new configuration

(c.) Call a bin a singleton if it has exactly one ball contained in it. What is the expected number of singletons at time T?

(d.) Suppose that your friend randomly chooses the setting on the switch so that the situation is equally likely to be that of Figure 3 and Figure 4. What is the variance of the total number of distinct balls in B_1 and B_2 after 1 second?

(e.) Alice and Bob decide to play a new game. They flip the switch so that the set up is as in Figure 4, but with a twist. Now, the balls in $f_{(1)}$ arrive according to a Poisson process with rate λ, but all other sets of balls ($f_{(2)}, f_{(3)}, \dots$) arrive according to a Poisson process with rate 2λ. The game is as follows: Alice wins if at any point in time, B_3 contains 10 balls more than B_2; Bob wins if at any point in time, B_2 contains 10 balls more than B_3 (the game is over when either Alice or Bob wins). What is the probability that Alice wins?

END OF THE EXAM.

Please check whether you have written your name and SID on every page.