Midterm Exam

Last Name	First Name	SID

Rules.

- You have 80 minutes (12:40pm - 2:00pm) to complete this exam.
- The maximum you can score is 120.
- The exam is not open book, but you are allowed one side of a sheet of handwritten notes; calculators will be allowed. No phones.
- No form of collaboration between the students is allowed. If you are caught cheating, you may fail the course and face disciplinary consequences.

Please read the following remarks carefully.

- Show all work to get any partial credit.
- Take into account the points that may be earned for each problem when splitting your time between the problems.

Problem	points earned	out of
Problem 1		40
Problem 2		20 + 5
Problem 3		20
Problem 4		20
Problem 5		20
Problem 6		20
Total		140 + 5 (Bonus)

Problem 1 [40]

(a) [15] For the Markov chain X_n with the state transition diagram shown in Figure 1, let $T_3 = \min\{n \geq 0 \mid X_n = 3\}$. Assume that X_0 is uniformly distributed in $\{0, 1, 2, 3\}$. Find $E(T_3)$.

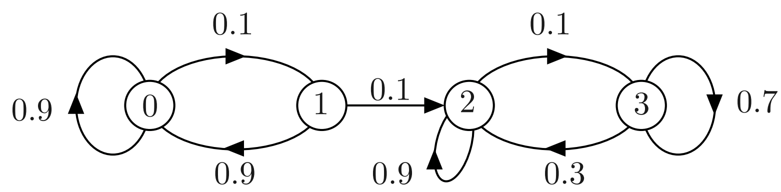

Figure 1: State transition diagram for Problem 1.

(b) [5] What are all the invariant distributions of the Markov chain shown in Figure 1?

(c) [10] For the same Markov chain, let $T = \max\{n \geq 0 \mid X_n \leq 1\}$. Find $E[T \mid X_0 = 0]$.

(d) [5] For the same Markov chain, what are the long term fractions of time that X_n is in the four different states?

(e) [5] For the same Markov chain, assume that $\pi_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$. What can you say about π_n as $n \to \infty$?

Problem 2 [20 + 5 (bonus)] You have 10 quarters in your left pocket and 10 quarters in your right pocket. At each step, you choose one of the two pockets, with equal probabilities, and you remove a quarter from that pocket and put it in the other pocket. What is the average number of steps until you reach into an empty pocket?

(a) [10] Formulate the problem as a Markov chain hitting time.

(b) [10] Writing down the first step equations.

(c) [5] Bonus Solve the equations.

Problem 3 **[20]** Consider the hidden Markov chain $\text{HMC}(\pi_0, P, Q)$ with $\pi_0 = [0.4, 0.6]$ and

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.5 & 0.5 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} 0.5 & 0.5 \\ 0.6 & 0.4 \end{bmatrix}.$$

Recall that Q is the emission matrix, so $Q(x, y) = P[Y_0 = y \mid X_0 = x]$ for all x, y. Find $\text{MAP}[X_0, X_1 \mid Y_0 = 0, Y_1 = 1]$.

(a) **[5]** Explain your approach clearly and concisely.

(b) **[10]** Show your calculations clearly.

(c) **[5]** State your result.

Problem 4 [20] Let X_n be a Markov chain on $\{0,1\}$ with $P(X_0 = 0) = 0.5$ and $P(0,1) = P(1,0) = 0.5$. Also, let Y_n be an independent Markov chain on $\{0,1\}$ with $P(Y_0 = 0) = 0.5$ and $P(0,1) = P(1,0) = 0.01$. Finally, let $Z_n = X_n + Y_n$ for $n \geq 0$. Prove or disprove that $\{Z_n, n \geq 0\}$ is a Markov chain.

(a) [10] Explain your approach clearly and concisely.

(b) [10] Show your calculations clearly.

Problem 5 [20] Customers enter a burger chain restaurant following a Poisson process with rate 100. Every customer gets their burger "Animal Style" with probability p. We know that 500 customers arrive in the first 5 hours of the day.
In answering the following questions state your reasoning as clearly as you can.

(a) [10] Find the probability that n customers ordered their burger "Animal Style" in the first 5 hours.

(b) **[10]** Find the probability that n customers ordered their burger animal style in the first 2 hours.

Problem 6 **[20]** Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. Bernoulli random variables with mean p. Let also $Y_n = \sum_{m=1}^{n} X_m$ for $n \geq 1$. Find $P[Y_7 = 1 \mid Y_{10} = 1]$.

(a) **[10]** Explain your approach clearly and concisely.

(b) **[10]** Show your calculations.

(b) [10] Find the probability that n customers ordered their burger animal style in the first 2 hours.

Problem 6 [20] Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. Bernoulli random variables with mean p. Let also $Y_n = \sum_{m=1}^{n} X_m$ for $n \geq 1$. Find $P[Y_7 = 1 \mid Y_{10} = 1]$.

(a) [10] Explain your approach clearly and concisely.

(b) [10] Show your calculations.