13

Math 1A (Fall 2017) Midterm II (Thursday October 26, 3:40-5:00)

Name: [redacted] SID: [redacted]

Please write clearly and legibly. **Justify your answers** except Problem 1. Partial credits may be given to Problems 2, 3, 4, 5, and 6. When submitting the exam, insert scratch paper in it if you put solutions there or if you think you may get partial credit from scratch work. In that case write your name on scratch paper to avoid a mix-up.

1. Mark each of the following True (T) or False (F). No justification is necessary. (For each sub-problem, correct = 4 pts, no response = 2 pts, wrong = 0 pts.)

✓(1) (F) Let f and g be functions on $\mathbb{R}$. If $f(2) = 3$ and $\lim_{x\to 1} g(x) = 2$ then $\lim_{x\to 1} f(g(x)) = 3$.

✓(2) (F) Let f be a function defined on $\mathbb{R}$. A horizontal asymptote of the graph $y = f(x)$, if it exists, never intersects the graph $y = f(x)$.

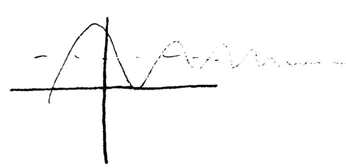
✓(3) (T) Let f be an even function which is differentiable everywhere. Then $f'(c) = 0$ for some number c.

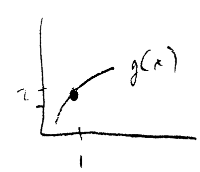
✓(4) (T) Let c be a constant. The function $f(x) = \sin x - 2x + c$ has a root regardless of the value of c.

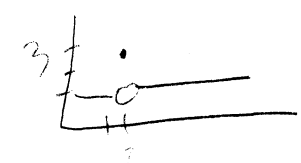
✓(5) (F) Let $f(x) = \begin{cases} g(x), & x < 0 \\ h(x), & x \geq 0 \end{cases}$ for differentiable functions $g(x)$ and $h(x)$.
Then $f'(0)$ exists as long as $g(0) = h(0)$.

$f(2)=3$

Even means can be flipped about y-axis



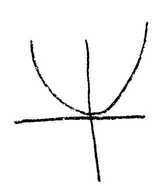




∴ Must ↓ then ↑

or

↑ then ↓



$f(x) = \sin x - 2x + C$

$f'(x) = \cos x - 2 = 0$

$1 - 2 = -1$

$\cos x - 2$ will never $= 0$

Always decreasing ∴ No crit points

At $f(0) = 0 - 0 + C$

✓2. (20 pts) Compute the following.

(1) (6 pts) $\lim_{x\to\infty} \frac{\sqrt{1+9e^{2x}}}{7-2e^x}$ (2) (6 pts) $\frac{d}{dx}\tan^{-1}(x^2)$

(3) (8 pts) The equation of the tangent line to the curve $x^{1/2}+y^{1/2}=5$ at $(4,9)$.

1.) $\lim_{x\to\infty}\left(\frac{\sqrt{1+9e^{2x}}}{7-2e^{x}}\cdot\frac{\frac{1}{e^x}}{\frac{1}{e^x}}\right)$

$= \lim_{x\to\infty}\left(\frac{\frac{\sqrt{1+9e^{2x}}}{e^x}}{\frac{7}{e^x}-\frac{2\cancel{e^x}}{\cancel{e^x}}}\right)$

$= \lim_{x\to\infty}\left(\frac{\frac{\sqrt{1+9e^{2x}}}{\sqrt{e^{2x}}}}{\frac{7}{e^x}-2}\right)$

* e^x is always positive *

$\therefore$ e^{2x} is always positive

$= \lim_{x\to\infty}\left(\frac{\sqrt{\frac{1}{e^{2x}}+\frac{9\cancel{e^{2x}}}{\cancel{e^{2x}}}}}{\frac{7}{e^x}-2}\right)$

* $\lim_{x\to\infty}\frac{1}{e^x}=0$

* $\lim_{x\to\infty}\frac{1}{e^{2x}}=0$

$= \frac{\sqrt{0+9}}{0-2} = \frac{3}{-2} = \boxed{\frac{-3}{2}}$

2.) $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$

$\therefore$

$\frac{d}{dx}\tan^{-1}(x^2) = \frac{1}{1+(x^2)^2}\cdot 2x$

$= \frac{1}{1+x^4}\cdot 2x$

$= \boxed{\frac{2x}{1+x^4}}$

3.) $x^{1/2}+y^{1/2}=5$

$\frac{1}{2}x^{-1/2}+\frac{1}{2}y^{-1/2}\cdot\frac{dy}{dx}=0$

$\frac{dy}{dx} = \frac{-\frac{1}{2}x^{-1/2}}{\frac{1}{2}y^{-1/2}} = \frac{-\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}$

$\frac{dy}{dx} = \frac{-2\sqrt{y}}{2\sqrt{x}}$

$\frac{dy}{dx}$ at $(4,9) = \frac{-2\sqrt{9}}{2\sqrt{4}} = \frac{-2(3)}{2(2)} = \frac{-3}{2}$

slope

point-slope form:

$y-9 = -\frac{3}{2}(x-4)$

$y-9 = -\frac{3}{2}x+6$

$\boxed{y = -\frac{3}{2}x+15}$

✓3. (15 pts) An ice cube is melting. (The shape of the ice remains to be a cube at every moment.) Its surface area is decreasing at 10cm^2 per second. When an edge is 30 cm long, how fast is the volume decreasing?

$V = l^3$

$SA = 6l^2$

$$\frac{dSA}{dt} = 12l \cdot \frac{dl}{dt}$$

$$-10 = 12(30) \cdot \frac{dl}{dt}$$

$$-10 = 360 \cdot \frac{dl}{dt}$$

$$\cancel{\frac{dV}{dt} = 3l^2}$$

$$\frac{dl}{dt} = \frac{-10}{360} = \frac{-1}{36}$$

$$\frac{dV}{dt} = 3l^2 \cdot \frac{dl}{dt}$$

$$\frac{dV}{dt} = 3(30)^2 \cdot \left(\frac{-1}{36}\right)$$

$$= \cancel{3}(900) \cdot \frac{-1}{\cancel{36}_{12}}$$

$$= \frac{-900}{12} = -75$$

$$\frac{dV}{dt} = -75 \text{ cm}^3/\text{s}$$

Volume is decreasing at 75 cm^3/s

4. (15 pts) Find all critical numbers of $f(x) = x^{x^2}$ on $(0, \infty)$. Determine at each critical number whether f has a local maximum, a local minimum, or neither.

$y = x^{x^2}$

$\ln y = x^2 \ln x$

$\frac{1}{y} \cdot \frac{dy}{dx} = x^2 \cdot \frac{1}{x} + \ln x (2x)$

$\frac{dy}{dx} = (x + 2x\ln(x))\, x^{x^2}$

$(x + 2x\ln(x))\, x^{x^2} = 0$ → discard b/c $x=0$ not in domain ($x^{x^2} = 0$ when $x=0$)

$x + 2x\ln(x) = 0$

discard ← $x(1 + 2\ln(x)) = 0$
b/c $x=0$ not in domain

$1 + 2\ln(x) = 0$

$2\ln(x) = -1$

$\ln(x) = -\frac{1}{2}$

$x = e^{-1/2} = \frac{1}{e^{1/2}}$

$\frac{1}{e} < \frac{1}{\sqrt{e}}$

$\left(\frac{1}{e}\right)^{(1/e)^2}$ is always positive

$\frac{1}{e} + 2\left(\frac{1}{e}\right)\ln e^{-1}$

$\frac{1}{e} - \frac{2}{e} = \frac{-1}{e}$ ← negative

thus $\frac{dy}{dx}$ is ⊖

$10 > \frac{1}{\sqrt{e}}$

10^{10^2} is always ⊕

$10 + 2(10)\ln(10)$ is always ⊕

Thus $\frac{dy}{dx}$ is ⊕

⊖ ⊕

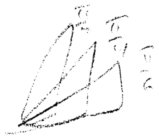
$0 \quad \frac{1}{e^{1/2}} \quad 10{,}000$

$\frac{dy}{dx}$ at $x = .0001$ is negative. Thus f(x) is decreasing from $(0, 1/e^{1/2})$

$\frac{dy}{dx}$ at $x = 10$ is positive. Thus f(x) is increasing from $(1/e^{1/2}, \infty)$

∴ $\boxed{\text{at } x = e^{-1/2}, \text{ there is a local minimum}}$

√5. (15 pts) For the function $f(x) = \sin^2 x + \cos x$ on $[0, \pi]$, find the *absolute* maximum and *absolute* minimum on $[0, \pi]$.



$$f(x) = (\sin x)^2 + \cos x$$

$$f'(x) = 2\sin(x)\cos(x) - \sin(x) = 0$$

$$\sin(x)\,(2\cos(x) - 1) = 0$$

$$\sin(x) = 0 \qquad 2\cos(x) - 1 = 0$$

$$x = 0, \pi \qquad \cos(x) = \frac{1}{2}$$

$$x = \frac{\pi}{3}$$

$$f(0) = 0 + 1 = 1$$

$$f(\pi) = (-1)^2 + 0 = 1$$

$$f(\pi/3) = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$$

Absolute Maximum: $\frac{5}{4}$ $(x = \pi/3)$

Absolute Minimum: 1 $(x = 0, \pi)$

√6. (15 pts) Using the precise definition of limits (a.k.a. the ϵ-δ definition), prove one of the three statements in the following list. (Even if you work on more than one, you get credit for *only one* of them.) • ~~$\lim_{x\to 2} 1/x = 1/2$.~~ • ~~$\lim_{x\to 0^+} \ln x = -\infty$.~~ • $\lim_{x\to 3}(x^2 - 6x + 12) = 3$.

Preliminary Analysis:

$\lim_{x\to 3} (x^2 - 6x + 12) = 3$

for every $\varepsilon > 0$, there is a $\delta > 0$ such that

if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$

$a = 3,\ L = 3,\ f(x) = x^2 - 6x + 12$

if $0 < |x-3| < \delta$ then $|(x^2 - 6x + 12) - 3| < \varepsilon$

if $0 < |x-3| < \delta$ then $|x^2 - 6x + 9| < \varepsilon$

if $0 < |x-3| < \delta$ then $|(x-3)^2| < \varepsilon$

if $0 < |x-3| < \delta$ then $x - 3 < \sqrt{\varepsilon}$ * true b/c $\varepsilon > 0$, given as condition *

this suggests that $\delta = \sqrt{\varepsilon}$

Proof:

Given $\varepsilon > 0$ and $\delta = \sqrt{\varepsilon}$. If $0 < |x-3| < \delta$, then...

$|x-3| < \sqrt{\varepsilon}$

$= |(x-3)^2| < \varepsilon$

$= |x^2 - 6x + 9| < \varepsilon$

$= |(x^2 - 6x + 12) - 3| < \varepsilon$

Thus, if $0 < |x-3| < \delta$, then $|(x^2 - 6x + 12) - 3| < \varepsilon$

By Definition of a limit, $\lim_{x\to 3} (x^2 - 6x + 12) = 3$