Math 53 First Midterm Exam, Prof. Srivastava
February 22, 2018, 5:10pm–6:30pm, 155 Dwinelle Hall.

Name: Nikhil Srivastava

SID: ______________________

GSI: ______________________

Name of the student to your left: ______________________
Name of the student to your right: ______________________

Instructions: Write all answers clearly in the provided space. This exam includes some space for scratch work at the bottom of pages 2 and 4 which will not be graded. Do not under any circumstances unstaple the exam. Write your name and SID on every page. Show your work — numerical answers without justification will be considered suspicious and will not be given full credit. Calculators, phones, cheat sheets, textbooks, and your own scratch paper are not allowed.

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Question	Points
1	10
2	10
3	13
4	10
5	21
6	10
7	10
8	16
Total:	100

Do not turn over this page until your instructor tells you to do so.

Name and SID: ____________________

1. (10 points) Find a parametric equation of the line defined by the intersection of the planes:

$$x - y + 3z = 5$$

$$3x + y - z = 3.$$

Let's choose y=t as the parameter since it appears simply in both equations. We then have:

$$x + 3z = 5 + t$$
$$3x - z = 3 - t$$

$$4x + 2z = 8 \implies z = 4 - 2x$$

$$\implies 3x - (4 - 2x) = 3 - t$$

$$\implies 5x = 7 - t \implies x = \frac{7}{5} - \frac{t}{5}$$

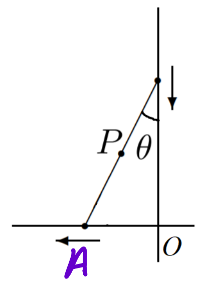
$$\implies z = 4 - \frac{14}{5} + \frac{2t}{5} = \frac{2t}{5} + \frac{6}{5}$$

[Scratch Space Below]

So an equation is:

$$\underline{\vec{r}(t) = \langle -t/5 + 7/5,\ t,\ \tfrac{2t}{5} + 6/5 \rangle}$$

2. The top extremity of a ladder of unit length rests against a vertical wall, while the bottom is being pulled away, as shown below.



(a) (8 points) Find a parameterized curve $\mathbf{r}(\theta)$ tracing the trajectory of the midpoint P of the ladder as it goes from fully vertical to horizontal, using as parameter the angle θ between the ladder and the vertical wall, and treating the point at which the wall and the ground meet as the origin.

By trigonometry, because the hypotenuse has length one we have:

$|\vec{OA}| = \sin(\theta) \implies \vec{OA} = \langle -\sin\theta, 0 \rangle$

$\vec{AP} = \frac{1}{2}\langle \sin\theta, \cos\theta \rangle$ since $|\vec{AP}| = 1/2$

Thus, the point P has position vector $\vec{OP} = \vec{OA} + \vec{AP}$

$= \langle -\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta \rangle$

and the param. curve is $\bar{r}(\theta) = \langle -\frac{1}{2}\sin\theta, \frac{1}{2}\cos\theta \rangle$

(b) (2 points) Is the speed of P (as a function of θ) increasing, decreasing, or constant as θ varies from 0 to $\pi/2$?

The velocity is $\bar{r}'(\theta) = \langle -\frac{1}{2}\cos\theta, -\frac{1}{2}\sin\theta \rangle$

So the speed is $|\bar{r}'(\theta)| = \sqrt{\frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta}$

$= 1/2$, which is constant.

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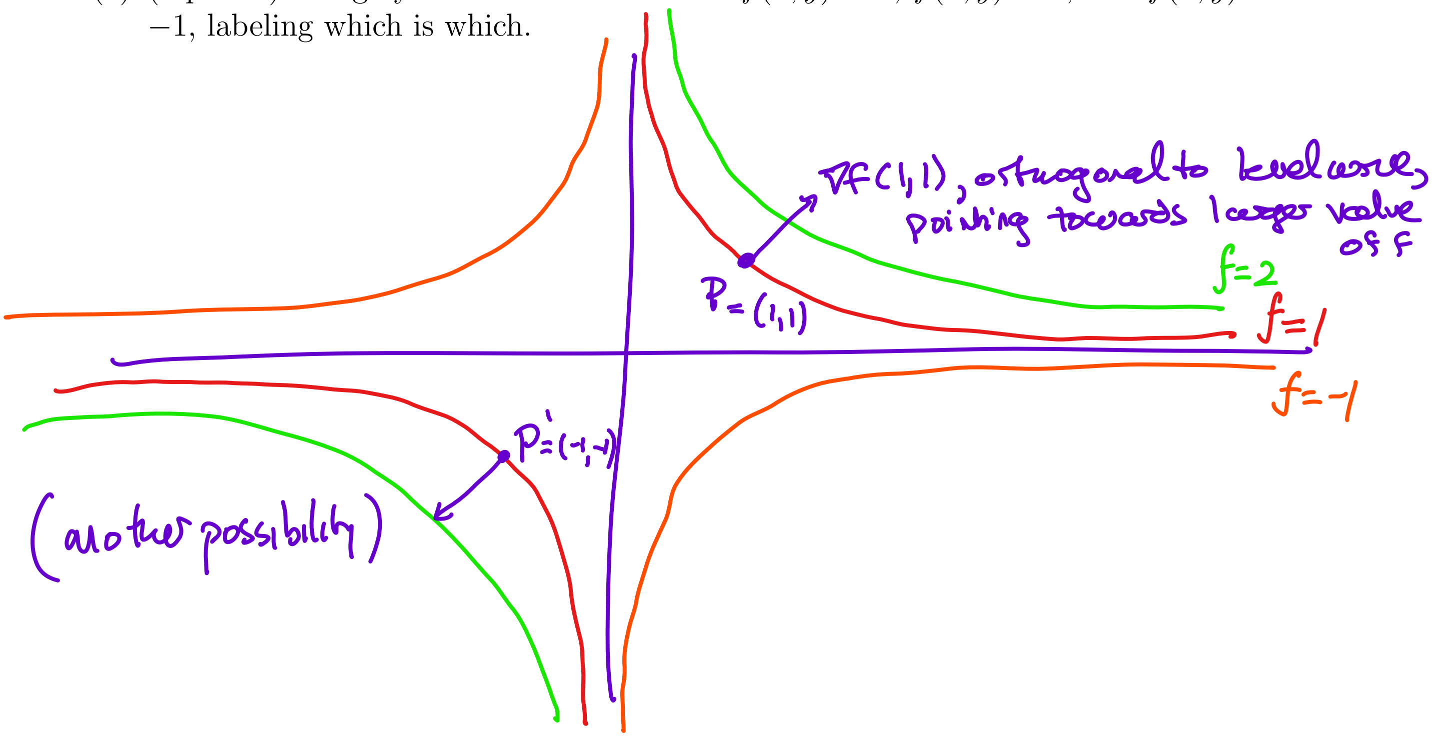
3. Consider the function $f(x, y) = xy$.

(a) (5 points) Roughly sketch the level curves $f(x, y) = 1$, $f(x, y) = 2$, and $f(x, y) = -1$, labeling which is which.



(b) (5 points) Find a point P on the curve $f(x, y) = 1$ at which the directional derivative along the direction $\mathbf{u} = \frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$ is equal to zero.

$D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \langle 1/\sqrt{2}, -1/\sqrt{2} \rangle = \frac{f_x}{\sqrt{2}} - \frac{f_y}{\sqrt{2}}$

is zero exactly when $f_x = f_y$, i.e.

$f_x = y = x = f_y$.

There are <u>two</u> such points on $\{f=1\}$, namely

$(1,1)$ and $(-1,-1)$.

(c) (3 points) Sketch the direction of the vector ∇f at the point P that you found above, originating from P in the drawing in part (a).

4. (10 points) The parameterized curves $\mathbf{a}(t) = \langle 3t, t^2 - 2, e^t + 1\rangle$ and $\mathbf{b}(s) = \langle s^2 - s, -2s, s^3 + s^2\rangle$ intersect at the point $P = (0, -2, 2)$. Find the cosine of the (acute) angle of their intersection at P.

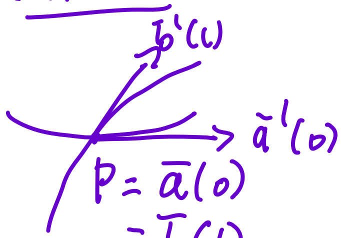
Inspecting the first coordinate, $P = \bar{a}(0)$

Inspecting the second coordinate, $P = \bar{b}(1)$.

The velocities at P are:

$$\bar{a}'(0) = \langle 3, 2t, e^t\rangle\Big|_{t=0} = \langle 3, 0, 1\rangle$$

Cartoon!



$$\bar{b}'(0) = \langle 2s-1, -2, 3s^2 + 2s\rangle\Big|_{s=1} = \langle 1, -2, 5\rangle$$

$$\text{So } \cos(\theta) = \frac{\bar{a}'(0) \cdot \bar{b}'(1)}{|\bar{a}'(0)|\,|\bar{b}'(1)|} = \frac{3 + 0 + 5}{\sqrt{9+1}\sqrt{1+4+25}}.$$

$$= \frac{15}{\sqrt{10}\sqrt{30}} = \frac{15}{2\sqrt{10}}$$

(which corresponds to an acute angle since > 0)

[Scratch Space Below]

5. Consider the function $f(x, y) = \sqrt{e^x + 3e^{y-1}}$.

(a) (5 points) Find the total differential of f at $x = 0, y = 1$.

$$df = f_x dx + f_y dy = \frac{e^x}{2\sqrt{e^x+3e^{y-1}}}dx + \frac{3e^{y-1}}{2\sqrt{e^x+3e^{y-1}}}dy$$

So at $(0,1)$

$$df = \frac{1}{2\sqrt{1+3}}dx + \frac{3}{2\sqrt{1+3}}dy$$

$$= \underline{\underline{\frac{1}{4}dx + \frac{3}{4}dy}}$$

(b) (5 points) Find the equation of the tangent plane to the graph $\{(x, y, z) : z = f(x, y)\}$ of f at the point $P = (0, 1, f(0, 1))$.

The differential gives the equation of the tangent plane which is:

$$(z - \underbrace{2}_{f(0,1)}) = \frac{1}{4}(x-0) + \frac{3}{4}(y-1)$$

which after simplification is

$$\frac{1}{4}x + \frac{3}{4}y - z = -5/4$$

(c) (5 points) Use the equation from (b) to compute an approximation to the value of $f(.01, .99)$.

$$f(.01, .99) = f(0 + .01, 1 - .01)$$

$$\approx f(0,1) + \frac{1}{4}(.01) + \frac{3}{4}(-.01)$$

$$= 2 - \frac{2}{4}.01 = \underline{\underline{1.995}}.$$

(d) (3 points) Is there a point Q on the graph at which the tangent plane to the graph is horizontal (i.e., parallel to the xy-plane)? If so, find such a point, and if not explain why.

For the graph to be horizontal at Q, we would need $f_x = f_y = 0$ (i.e, (x,y) a critical pt).

However in this case, $f_x = \dfrac{e^x}{2\sqrt{e^x + 3e^{y-1}}} > 0$, $f_y = \dfrac{3e^{y-1}}{2\sqrt{e^x + 3e^{y-1}}} > 0$ for all $(x,y) \in \mathbb{R}^2$, so there is <u>no such point</u>.

(e) (3 points) Is there a point R on the graph where the tangent plane is vertical (i.e., perpendicular to the xy-plane)? If so, find such a point, and if not explain why.

The graph of the function is the level surface of $F(x,y,z) = f(x,y) - z = 0$. To have a vertical tangent plane, $\nabla F(x,y,z)$ would need to have a zero in the last component, which is impossible since $\nabla F(x,y,z) = \langle f_x, f_y, -1 \rangle$.

More conceptually, the graph of a function cannot have a vertical tangent plane since they would fail the vertical line test.

6. (10 points) Let $w = w(x, y)$ be a differentiable function and let $x = s^2t$ and $y = s + 1/t$. Use the chain rule to express the partial derivatives w_s and w_t in terms of w_x, w_y, s, and t.

The chain Rule says:

$$w_s = w_x x_s + w_y y_s = w_x (2st) + w_y (1) = \underline{\underline{2stw_x + w_y}}$$

$$w_t = w_x x_t + w_y y_t = w_x s^2 + w_y\left(-\frac{1}{t^2}\right) = \underline{\underline{s^2 w_x - \frac{w_y}{t^2}}}.$$

7. (10 points) Suppose $z(x, y)$ is a differentiable function satisfying the equation

$$x^2 - y^2 + z^2 - 2z = 4.$$

Find $\partial z/\partial x$ and $\partial^2 z/\partial x^2$ in terms of x, y, z.

Letting $z = z(x,y)$ and treating y as a constant, we have:

$$0 = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial x}(z^2 - 2z) =$$

$$= 2x - 0 + 2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x} \implies \frac{\partial z}{\partial x} = \frac{2x}{2-2z} = \frac{x}{1-z} //$$

Implicitly Differentiating again, we have

$$0 = \frac{\partial}{\partial x}(2x) - \frac{\partial}{\partial x}\left(2z\frac{\partial z}{\partial x} - 2\frac{\partial z}{\partial x}\right) = 2 - \left[2z\frac{\partial^2 z}{\partial x^2} + 2\left(\frac{\partial z}{\partial x}\right)^2 - 2\frac{\partial^2 z}{\partial x^2}\right]$$

So $$\frac{\partial^2 z}{\partial x^2} = \frac{2 - 2\left(\frac{\partial z}{\partial x}\right)^2}{2z-2} = \left[2 - 2\left(\frac{2x}{2-2z}\right)^2\right] \Big/ 2z-2 = \frac{1 - \left(\frac{x}{1-z}\right)^2}{z-1} //$$

8. Let $A(x, y)$ be the area of the triangle with vertices at the points $P = (0, 0), Q = (1, 2)$, and $R = (x, y)$.

(a) (5 points) Derive a formula for $A(x, y)$.

$A(x,y)$ is half the area of the parallelogram spanned by $\overrightarrow{PQ} = \langle 1,2 \rangle$ and $\overrightarrow{PR} = \langle x,y \rangle$, which is the area as the area spanned by $\langle 1,2,0 \rangle$ and $\langle x,y,0 \rangle$. By the cross product formula, this is:

$$\frac{1}{2}\left| \langle 1,2,0 \rangle \times \langle x,y,0 \rangle \right| = \frac{1}{2}\left| \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ x & y & 0 \end{vmatrix} \right| = \frac{1}{2}\left| (y-2x)\hat{k} \right|$$

$$= \underline{\underline{\frac{1}{2}|y-2x|}}.$$

(b) (5 points) Identify all critical points of $\overset{2}{A}(x, y)$, and classify them as local minima, maxima, or saddle points, with justification.

We have $A^2(x,y) = \frac{1}{4}(y-2x)^2$.

Its partial derivatives are: $\frac{\partial}{\partial x}A^2 = \frac{-2 \cdot 2}{4}(y-2x)$

$$\frac{\partial A^2}{\partial y} = \frac{1}{4}2(y-2x),$$

which are zero precisely on the line $\underline{\underline{y=2x}}$.

Since $A^2 \geq 0$ and $A^2 = 0$ on this line, all these points are local minima.

(c) (6 points) Use Lagrange multipliers to find a point $R = (x, y)$ on the curve

$$xy = -1$$

which minimizes $A(x, y)$, and determine the minimum area. (hint: it might be easier to minimize A^2)

Let $f(x,y) = A^2 = \frac{1}{4}(y-2x)^2$, $g(x,y) = xy$.

The Lagrange multiplier equations are:

① — $f_x = -(y-2x) = \lambda g_x = \lambda y$

② — $f_y = \frac{1}{2}(y-2x) = \lambda g_y = \lambda x$

and $xy = -1$. ③

① and ② $\Rightarrow$ $2\lambda x = -\lambda y \Rightarrow \lambda = 0$ or $2x = -y$

(impossible, since it would imply $y = 2x \Rightarrow A = 0$)

So $\frac{-y}{2} \cdot y = -1 \longrightarrow y^2 = 2 \Rightarrow y = +\sqrt{2}$ or $-\sqrt{2}$

$x = -1/\sqrt{2}$ or $1/\sqrt{2}$

So the optima must be $(-1/\sqrt{2}, \sqrt{2})$ and $(1/\sqrt{2}, -\sqrt{2})$

at which $A = \frac{1}{2}|\sqrt{2}+\sqrt{2}| = \underline{\underline{\sqrt{2}}}$.