Math 54	**Name:**	____________________
Fall 2017		
Final Exam	**Student ID**:	____________________
12/14/17		
Time Limit: 170 Minutes	**GSI or Section**:	____________________

This exam contains 9 pages (including this cover page) and 10 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

This is a closed book exam. No notes or calculators are permitted.
We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

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1. (10 points) Consider the 2×2 matrix

$$M_a = \begin{bmatrix} a & 2-a \\ 2+a & -a \end{bmatrix}.$$

(a) (2 points) Find all real values of a such that M_a is invertible.

Solution: The determinant of M_a is $a(-a) - (2-a)(2+a) = -4$, so M_a is invertible for any value of a.

(b) (2 points) Find all real values of a such that M_a is diagonalizable.

Solution: The characteristic polynomial of M_a is $\chi_{M_a}(\lambda) = \lambda^2 - 4$, which has two distinct real roots, so M_a is diagonalizable for all a.

(c) (4 points) Find **all** eigenvectors of M_1.

Solution: We found in (b) that the eigenvalues of M_1 were ± 2, so we must find a basis for $\mathrm{Nul}(M_1 - 2I)$ and $\mathrm{Nul}(M_1 + 2I)$.
We have

$$E_{-2} = \mathrm{Nul}(M_1 + 2I) = \mathrm{Nul}\left(\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} 3 & 1 \\ 0 & 0 \end{bmatrix}\right),$$

which is the set of vectors $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $3x_1 = x_2$. Thus, an eigenvector for $\lambda = -2$ is $\mathbf{v}_{1,-2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.
Similarly,

$$E_2 = \mathrm{Nul}(M_1 - 2I) = \mathrm{Nul}\left(\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}\right) = \mathrm{Nul}\left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\right),$$

so an eigenvector for $\lambda = 2$ is $\mathbf{v}_{1,2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Thus, the eigenvectors of M_1 are the vectors of the form $c\mathbf{v}_{1,-2}$ and the vectors of the form $c\mathbf{v}_{1,2}$, where $c \in \mathbb{R}$ is nonzero.

(d) (2 points) Is M_1 orthogonal?

Solution: No.
By definition, M_1 is orthogonal if $M_1^T M_1 = I$. We compute:

$$\begin{aligned} M_1^T M_1 &= \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} \end{aligned}$$

which is not the identity matrix so M_1 is not orthogonal.
Alternatively, we found in part (a) that the determinant of M_a is always -4. An orthogonal matrix has determinant ± 1 so M_a can never be orthogonal.

2. (10 points) In this problem, you will compute a singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$$

That is, you will find a 3×2 matrix Σ with nonnegative entries whose only nonzero entries are Σ_{11} and Σ_{22}, a 3×3 orthogonal matrix U, and a 2×2 orthogonal matrix V such that $A = U\Sigma V^T$.

(a) (3 points) Find the matrix Σ.

Solution: The diagonal entries of Σ are the square roots of the eigenvalues of $A^T A$. We compute

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}$$

which has characteristic equation $\lambda^2 - 15\lambda$ so the eigenvalues of $A^T A$ are 15 and 0. Thus, we have

$$\Sigma = \begin{bmatrix} \sqrt{15} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) (3 points) Find the matrix V.

Solution: The columns of V are an orthonormal basis of eigenvectors of $A^T A$. An eigenvector for the eigenvalue $\lambda = 15$ is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and an eigenvector for the eigenvalue $\lambda = 0$ is $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Normalizing, we have

$$V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

(c) (4 points) Find the matrix U.

Solution: The columns of U come from extending $\{Av_i/\sigma_i\}$ where σ_i are the nonzero singular values of A to an orthonormal basis of $\mathbb{R}^3$. Thus, the first column of U must be

$$\frac{1}{\sqrt{15}} A \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and we need to extend this to any orthonormal basis of $\mathbb{R}^3$. Thus, we can take

$$U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -\sqrt{2/3} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}$$

among infinitely many other possibilities.

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

   (a) (2 points) Suppose A is an $n \times n$ matrix. The linear homogeneous system of equations $x'(t) = Ax(t)$ has n linearly independent solutions only when A is diagonalizable.

   Solution: False.
   The system of equations always has n linearly independent solutions. For example, the columns of e^{At} are one such set of solutions.

   (b) (2 points) Suppose that A is a symmetric $n \times n$ matrix and W is a subspace of $\mathbb{R}^n$ such that $Aw \in W$ for all $w \in W$. Then, $Av \in W^\perp$ for all $v \in W^\perp$ where $W^\perp$ is the orthogonal complement of W with respect to the dot product.

   Solution: True.
   If $v \in W^\perp$ and $w \in W$, then $Av \cdot w = v \cdot A^T w = v \cdot Aw = 0$ since $Aw \in W$. Thus, $Av \in W^\perp$ since it is orthogonal to every vector in W.

   (c) (2 points) Suppose that $v_1(t), \ldots, v_n(t)$ are vector functions taking values in $\mathbb{R}^n$. If the Wronskian $W[v_1, \ldots, v_n](t)$ is equal to 0 for all $t \in \mathbb{R}$, then $v_1(t), \ldots, v_n(t)$ are linearly dependent.

   Solution: False.
   Consider $v_1(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2(t) = \begin{bmatrix} t \\ 0 \end{bmatrix}$. Then, the Wronskian always vanishes, but these functions are linearly independent as $c_1 v_1(t) + c_2 v_2 = 0$ implies that $c_1 + c_2 t = 0$ for all t which further implies $c_1 = c_2 = 0$ by plugging in $t = 0$ and $t = 1$.

   (d) (2 points) Every symmetric $n \times n$ matrix with real entries is similar to a diagonal matrix with real entries.

   Solution: True.
   This is a corollary of the spectral theorem, which says that every real symmetric matrix is orthogonally diagonalizable and has real eigenvalues.

   (e) (2 points) The set of solutions to $ay'' + by' + cy = 0$ is a two-dimensional vector space for any $a, b, c \in \mathbb{R}$.

   Solution: False.
   The statement is true when $a \neq 0$. However, we could take $a = b = 0$ and $c = 1$ and the only solution would be $y = 0$.

4. (10 points) Suppose that $\mathbb{P}_3$ is the vector space of polynomials of degree at most three and $E_3 \subset \mathbb{P}_3$ the set of even polynomials of degree at most three, i.e., E_3 consists of $p(x) \in \mathbb{P}_3$ such that $p(-x) = p(x)$.

(a) (3 points) Show that E_3 is a subspace of $\mathbb{P}_3$.

Solution: We must show that E_3 satisfies the three properties of a subspace. The zero polynomial is even so $0 \in E_3$. Suppose that $p, q \in E_3$. Then,

$$(p+q)(-x) = p(-x) + q(-x) = p(x) + q(x) = (p+q)(x)$$

so $p + q \in E_3$. Finally, if $p \in E_3$ and $c \in \mathbb{R}$, then

$$(cp)(-x) = cp(-x) = cp(x) = (cp)(x)$$

so $cp \in E_3$.

(b) (2 points) What is the dimension of E_3?

Solution: Suppose that $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \in E_3$. Then, $p(x) = p(-x)$ gives that

$$a_3x^3 + a_2x^2 + a_1x + a_0 = -a_3x^3 + a_2x^2 - a_1x + a_0$$

or

$$2a_3x^3 + 2a_1x = 0$$

which implies that $a_3 = a_1 = 0$ since x^3 and x are linearly independent. Thus, we have seen that $E_3 \subset \operatorname{span}\{1, x^2\}$. But, 1 and x^2 are both even so we have $E_3 = \operatorname{span}\{1, x^2\}$. Since 1 and x^2 are linearly independent, this shows that $\dim E_3 = 2$.

(c) (5 points) Compute the orthogonal projection of x to E_3 when $\mathbb{P}_3$ is equipped with the inner product $\langle p, q\rangle = \int_0^1 p(x)q(x)\,dx$.

Solution: We first need an orthogonal basis of E_3. We apply Gram-Schmidt to $\{1, x^2\}$. Then, we get that

$$u_1 = 1$$

$$u_2 = x^2 - \frac{\langle 1, x^2\rangle}{\langle 1, 1\rangle}1 = x^2 - \frac{1}{3}.$$

Then,

$$\operatorname{Proj}_{E_3}(x) = \frac{\langle 1, x\rangle}{\langle 1, 1\rangle}1 + \frac{\langle x^2 - 1/3, x\rangle}{\langle x^2 - 1/3, x^2 - 1/3\rangle}(x^2 - 1/3) = \frac{1}{2} + \frac{15}{16}(x^2 - 1/3)$$

$$= \frac{1}{16}(3 + 15x^2).$$

5. (10 points) Let A be an $n \times n$ matrix such that $A^k = 0$ for some $k \geq 1$.

(a) (4 points) Show that 0 is the only eigenvalue of A.

Solution: Suppose that λ is an eigenvalue of A with eigenvector v. Then, $Av = \lambda v$. Therefore,

$$0 = A^k v = \lambda^k v$$

so $\lambda^k = 0$ since $v \neq 0$. This implies that $\lambda = 0$.

(b) (2 points) Show that if A is diagonalizable then A is the zero matrix.

Solution: By part (a), we know that the only eigenvalue of A is 0. Thus, if A is diagonalizable, we would have

$$A = P0P^{-1} = 0.$$

(c) (4 points) Show that if $\dim \mathrm{Nul}(A^2) = \dim \mathrm{Nul}(A)$ then A is the zero matrix.

Solution: We claim that $\mathbb{R}^n = \mathrm{Nul}(A^k) = \mathrm{Nul}(A)$, which would imply $A = 0$. Suppose that $v \neq 0$ is such that $A^k v = 0$. Then, $A^{k-1}v = 0$ since $A^{k-2}v$ is a vector in $\mathrm{Nul}(A^2) = \mathrm{Nul}(A)$. If $k > 2$, apply the same argument to deduce that $A^{k-2}v = 0$. Repeating this gives that $A^{k-\ell}v = 0$ for any $\ell < k$. In particular, $Av = 0$. Thus, proves the claim as we have shown $\mathbb{R}^n = \mathrm{Nul}(A^k) \subset \mathrm{Nul}(A)$.

6. (10 points) Consider the following sets of vector functions.

$$(1)\ \left\{\begin{bmatrix}2e^t\\3e^{2t}\end{bmatrix},\begin{bmatrix}e^t\\e^{2t}\end{bmatrix}\right\}\qquad (2)\ \left\{\begin{bmatrix}2e^t\\2e^t\end{bmatrix},\begin{bmatrix}e^t\\e^t\end{bmatrix}\right\}\qquad (3)\ \left\{\begin{bmatrix}2e^t\\0\end{bmatrix},\begin{bmatrix}t\\t\end{bmatrix}\right\}$$

(a) (5 points) Decide whether each of the sets is linearly independent.

Solution: (1) and (3).

We first compute the Wronskians.

$$(1)\ -e^{3t}\qquad (2)\ 0\qquad (3)\ 2te^t$$

This shows immediately that (1) and (3) are linearly independent since there are values of t where their Wronskian does not vanish. The Wronskian computation itself is not enough to deduce that (2) is linearly dependent, but it is since $\begin{bmatrix}2e^t\\2e^t\end{bmatrix}-2\begin{bmatrix}e^t\\e^t\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$.

(b) (5 points) For each of the above sets, show that the two vector functions solve a single system $x'(t)=Ax(t)$ for a 2×2 matrix A or explain why this is not possible.

Solution: (1) and (2)

The Wronskian of a set of such solutions either vanishes for all t or does not vanish at all by the Wronskian lemma. Therefore, (3) cannot be a solution to such a system. However, (1) consists of solutions to

$$x'(t)=\begin{bmatrix}1&0\\0&2\end{bmatrix}x(t)$$

and (2) consists of solutions to

$$x'(t)=\begin{bmatrix}1&0\\0&1\end{bmatrix}x(t).$$

Note that these are just two possibilities for the matrices among infinitely many others.

7. (10 points) Find a general solution to the following system.

$$x'(t) = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-3t}$$

Solution: We first find a homogeneous solution. To do this, we first find the eigenvalues of A. The characteristic polynomial of A is $\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1)$ so the eigenvalues of A are -1 and -2. An eigenvector for the eigenvalue -1 is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and an eigenvector for the eigenvalue -2 is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Thus, the general homogeneous solution is

$$x_h(t) = C_1 e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Now, we'll find a particular solution using undetermined coefficients. We guess our particular solution has the form

$$x_p(t) = e^{-3t} \left(t \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \right).$$

Plugging into the equation, we get

$$e^{-3t} \left(t \begin{bmatrix} -3a_1 \\ -3a_2 \end{bmatrix} + \begin{bmatrix} a_1 - 3b_1 \\ a_2 - 3b_2 \end{bmatrix} \right) = e^{-3t} \left(t \begin{bmatrix} a_1 - 6a_2 + 2 \\ a_1 - 4a_2 + 1 \end{bmatrix} + \begin{bmatrix} b_1 - 6b_2 \\ b_1 - 4b_2 \end{bmatrix} \right)$$

That is, we need to solve $-3a_1 = a_1 - 6a_2 + 2, -3a_2 = a_1 - 4a_2 + 1, a_1 - 3b_1 = b_1 - 6b_2$, and $a_2 - 3b_2 = b_1 - 4b_2$. We do this by row reduction.

$$\left[\begin{array}{cccc|c} -4 & 6 & 0 & 0 & 2 \\ -1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -4 & 6 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Thus, we get that

$$x_p(t) = -te^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Finally, the general solution is

$$x(t) = x_h(t) + x_p(t) = C_1 e^{-t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - te^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} - e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

8. (a) (8 points) Find the general solution to the equation

$$y'' + 3y' + 2y = \frac{1}{e^t + 1}.$$

Solution: We first find a basis of solutions to the homogeneous equation $y''+3y'+2y = 0$. The auxiliary equation is $r^2+3r+2 = (r+2)(r+1) = 0$ so a basis of homogeneous solutions is $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$.

We will now use variation of parameters to find the general solution. We first compute

$$W[y_1, y_2](t) = \begin{vmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{vmatrix} = -e^{-3t}.$$

If we guess $y(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ and assume $v_1'(t)y_1(t) + v_2'(t)y_2(t) = 0$, we get that

$$\begin{bmatrix} v_1'(t) \\ v_2'(t) \end{bmatrix} = \frac{1}{W[y_1, y_2](t)} \begin{bmatrix} -y_2(t)f(t) \\ y_1(t)f(t) \end{bmatrix}$$

where $f(t)$ is the nonhomogeneous term. Thus, we have $v_1'(t) = \frac{e^t}{e^t+1}$ and $v_2'(t) = \frac{-e^{2t}}{e^t+1}$. Integrating, we get

$$v_1(t) = \int \frac{e^t}{e^t + 1}\, dt + C_1 = \int \frac{1}{u}\, du + C_1 = \ln(e^t + 1) + C_1$$

with $u = e^t + 1$, and

$$v_2(t) = -\int \frac{e^{2t}}{e^t + 1}\, dt + C_2 = -\int \frac{u - 1}{u}\, du + C_2 = -e^t - 1 + \ln(e^t + 1) + C_2$$

again with $u = e^t + 1$. Thus, a general solution is

$$y(t) = C_1 e^{-t} + C_2 e^{-2t} + e^{-t}\ln(e^t + 1) + e^{-2t}\ln(e^t + 1)$$

where we dropped the term $e^{-2t}(e^t+1) = e^{-t}+e^{-2t}$ since it is a homogeneous solution.

(b) (2 points) Find the solution to the differential equation from part (a) that satisfies $y(0) = 0$ and $y'(0) = -3\ln(2) + 1$.

Solution: We have

$$y'(t) = -C_1 e^{-t} - 2C_2 e^{-2t} - e^{-t}\ln(e^t + 1) + \frac{1}{e^t + 1} - 2e^{-2t}\ln(e^t + 1) + \frac{e^{-t}}{e^t + 1}$$

so the initial conditions give $C_1+C_2 = 2\ln(2)$ and $-C_1-2C_2-3\ln(2)+1 = -3\ln(2)+1$. This implies that $C_1 = 4\ln(2)$ and $C_2 = -2\ln(2)$. Thus, the solution is

$$y(t) = 4\ln(2)e^{-t} - 2\ln(2)e^{-2t} + e^{-t}\ln(e^t + 1) + e^{-2t}\ln(e^t + 1).$$

9. (a) (3 points) Find the Fourier cosine series for the function $f : [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = (\sin(x) + 1)^2 + (\cos(x) + 1)^2 - 2\sin(x)$$

Solution: Using the trigonometric identity $\cos^2(x) + \sin^2(x) = 1$ we have

$$f(x) = \sin^2(x) + 2\sin(x) + 1 + \cos^2(x) + 2\cos(x) + 1 - 2\sin(x) = 3 + 2\cos(x)$$

In other words, the Fourier cosine series of $f(x)$ has coefficients $a_0 = 3$, $a_1 = 2$, and $a_n = 0$ for $n \geq 2$.

(b) (4 points) Find a solution to the following initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0 \\
&\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \\
&u(x, 0) = (\sin(x) + 1)^2 + (\cos(x) + 1)^2 - 2\sin(x)
\end{aligned}$$

Solution: The general solution to the heat equation with Neumann boundary conditions is given by

$$\sum_{n=0}^{\infty} a_n e^{-n^2 t} \cos(nx)$$

In order for the solution to satisfy the given initial condition the a_n must coincide with the Fourier coefficients of $(\sin(x)+1)^2 + (\cos(x)+1)^2 - 2\sin(x)$, which were computed in part (a). It follows that the solution to the given initial value problem is

$$u(x, t) = 3 + 2e^{-t}\cos(x)$$

(c) (3 points) Find a solution to the following (nonhomogeneous) initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + t^2, \qquad 0 < x < \pi, \quad t > 0 \\
&\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \\
&u(x,0) = (\sin(x)+1)^2 + (\cos(x)+1)^2 - 2\sin(x)
\end{aligned}$$

Solution: By the superposition principle, it suffices to find a solution to the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + t^2$ with Neumann boundary conditions and initial data $u(x,0) = 0$. The inhomogeneous term does not depend on x so we should take u to only depend on t, that is, $u(x,t) = F(t)$. The equation then becomes $F'(t) = t^2$ with initial conditions $F(0) = 0$, so we get $u(x,t) = F(t) = t^3/3$. Therefore, a solution to the given initial value problem is

$$u(x,t) = 3 + 2e^{-t}\cos(x) + t^3/3$$

Alternatively, one can guess a solution of the form

$$u(x,t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\cos(nx) + b_n(t)\sin(nx)$$

then plug-in and solve to reach the same result.

10. (10 points) Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \pi - x & \text{if } 0 \leq x \leq \pi \\ \pi + x & \text{if } -\pi \leq x \leq 0 \end{cases}.$$

(a) (6 points) Compute the Fourier series of $f(x)$.

Solution: Since f is even, we have that $b_n = 0$ for all n. We then compute the cosine coefficients.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(nx)\, dx \\ &= \frac{2}{\pi} \left(\left[\frac{\pi}{n} \sin(nx) \right]_0^{\pi} - \left[\frac{x \sin(nx)}{n} \right]_0^{\pi} + \frac{1}{n} \int_0^{\pi} \sin(nx)\, dx \right) \\ &= \frac{2}{n^2 \pi} \left[-\cos(nx) \right]_0^{\pi} = \frac{2(1 - (-1)^n)}{\pi n^2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Therefore, the Fourier series of $f(x)$ is

$$F(x) = \frac{\pi}{2} + \sum_{k=0}^{\infty} \frac{4}{\pi (2k+1)^2} \cos((2k+1)x).$$

(b) (2 points) If F is the Fourier series that you computed in part (a), compute $F\left(\frac{9\pi}{4}\right)$.

Solution: Since f is piecewise-differentiable $F(x) = f(x)$ on $(-\pi, \pi)$. Moreover, $F(x)$ is 2π-periodic so

$$F\left(\frac{9\pi}{4}\right) = F\left(\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) = \frac{3\pi}{4}$$

(c) (2 points) Use the Fourier series that you computed in part (a) to compute the sum $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots$

Solution: By the convergence mentioned in the solution to the previous part, we have

$$\pi = F(0) = \frac{\pi}{2} + \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

Therefore,

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

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