Math 54
Fall 2017
Final Exam
12/14/17
Time Limit: 170 Minutes

Name: ______________________

Student ID: ______________________

GSI or Section: ______________________

This exam contains 9 pages (including this cover page) and 10 problems. Problems are printed on both sides of the pages. Enter all requested information on the top of this page.

> This is a closed book exam. No notes or calculators are permitted.
> We will drop your lowest scoring question for you.

You are required to show your work on each problem on this exam. **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.

If you need more space, there are blank pages at the end of the exam. Clearly indicate when you have used these extra pages for solving a problem. However, it will be greatly appreciated by the GSIs when problems are answered in the space provided, and your final answer **must** be written in that space. Please do not tear out any pages.

Do not write in the table to the right.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

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1. (10 points) Consider the 2×2 matrix

$$M_a = \begin{bmatrix} a & 2-a \\ 2+a & -a \end{bmatrix}.$$

(a) (2 points) Find all real values of a such that M_a is invertible.

(b) (2 points) Find all real values of a such that M_a is diagonalizable.

(c) (4 points) Find **all** eigenvectors of M_1.

(d) (2 points) Is M_1 orthogonal?

2. (10 points) In this problem, you will compute a singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$$

That is, you will find a 3×2 matrix Σ with nonnegative entries whose only nonzero entries are Σ_{11} and Σ_{22}, a 3×3 orthogonal matrix U, and a 2×2 orthogonal matrix V such that $A = U\Sigma V^T$.

(a) (3 points) Find the matrix Σ.

(b) (3 points) Find the matrix V.

(c) (4 points) Find the matrix U.

3. (10 points) Label the following statements as True or False. The correct answer is worth 1 point and a brief justification is worth 1 point. Credit for the justification can only be earned in conjunction with a correct answer. No points will be awarded if it is not clear whether you intended to mark the statement as True or False.

   (a) (2 points) Suppose A is an $n \times n$ matrix. The linear homogeneous system of equations $x'(t) = Ax(t)$ has n linearly independent solutions only when A is diagonalizable.

   (b) (2 points) Suppose that A is a symmetric $n \times n$ matrix and W is a subspace of $\mathbb{R}^n$ such that $Aw \in W$ for all $w \in W$. Then, $Av \in W^\perp$ for all $v \in W^\perp$ where $W^\perp$ is the orthogonal complement of W with respect to the dot product.

   (c) (2 points) Suppose that $v_1(t), \ldots, v_n(t)$ are vector functions taking values in $\mathbb{R}^n$. If the Wronskian $W[v_1, \ldots, v_n](t)$ is equal to 0 for all $t \in \mathbb{R}$, then $v_1(t), \ldots, v_n(t)$ are linearly dependent.

   (d) (2 points) Every symmetric $n \times n$ matrix with real entries is similar to a diagonal matrix with real entries.

   (e) (2 points) The set of solutions to $ay'' + by' + cy = 0$ is a two-dimensional vector space for any $a, b, c \in \mathbb{R}$.

4. (10 points) Suppose that $\mathbb{P}_3$ is the vector space of polynomials of degree at most three and $E_3 \subset \mathbb{P}_3$ the set of even polynomials of degree at most three, i.e., E_3 consists of $p(x) \in \mathbb{P}^3$ such that $p(-x) = p(x)$.

   (a) (3 points) Show that E_3 is a subspace of $\mathbb{P}_3$.

   (b) (2 points) What is the dimension of E_3?

   (c) (5 points) Compute the orthogonal projection of x to E_3 when $\mathbb{P}_3$ is equipped with the inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$.

5. (10 points) Let A be an $n \times n$ matrix such that $A^k = 0$ for some $k \geq 1$.

   (a) (4 points) Show that 0 is the only eigenvalue of A.

   (b) (2 points) Show that if A is diagonalizable then A is the zero matrix.

   (c) (4 points) Show that if $\dim \mathrm{Nul}(A^2) = \dim \mathrm{Nul}(A)$ then A is the zero matrix.

6. (10 points) Consider the following sets of vector functions.

$$(1)\ \left\{ \begin{bmatrix} 2e^t \\ 3e^{2t} \end{bmatrix}, \begin{bmatrix} e^t \\ e^{2t} \end{bmatrix} \right\} \qquad (2)\ \left\{ \begin{bmatrix} 2e^t \\ 2e^t \end{bmatrix}, \begin{bmatrix} e^t \\ e^t \end{bmatrix} \right\} \qquad (3)\ \left\{ \begin{bmatrix} 2e^t \\ 0 \end{bmatrix}, \begin{bmatrix} t \\ t \end{bmatrix} \right\}$$

(a) (5 points) Decide whether each of the sets is linearly independent.

(b) (5 points) For each of the above sets, show that the two vector functions solve a single system $x'(t) = Ax(t)$ for a 2×2 matrix A or explain why this is not possible.

7. (10 points) Find a general solution to the following system.

$$x'(t) = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{-3t}$$

8. (a) (8 points) Find the general solution to the equation

$$y'' + 3y' + 2y = \frac{1}{e^t + 1}.$$

(b) (2 points) Find the solution to the differential equation from part (a) that satisfies $y(0) = 0$ and $y'(0) = -3\ln(2) + 1$.

9. (a) (3 points) Find the Fourier cosine series for the function $f : [0, \pi] \to \mathbb{R}$ defined by

$$f(x) = (\sin(x) + 1)^2 + (\cos(x) + 1)^2 - 2\sin(x)$$

(b) (4 points) Find a solution to the following initial value problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \quad t > 0 \\
&\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \\
&u(x, 0) = (\sin(x) + 1)^2 + (\cos(x) + 1)^2 - 2\sin(x)
\end{aligned}$$

(c) (3 points) Find a solution to the following (nonhomogeneous) initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + t^2, \qquad 0 < x < \pi, \quad t > 0$$
$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$$
$$u(x,0) = (\sin(x)+1)^2 + (\cos(x)+1)^2 - 2\sin(x)$$

10. (10 points) Consider the function $f : [-\pi, \pi] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \pi - x & \text{if } 0 \leq x \leq \pi \\ \pi + x & \text{if } -\pi \leq x \leq 0 \end{cases}.$$

(a) (6 points) Compute the Fourier series of $f(x)$.

(b) (2 points) If F is the Fourier series that you computed in part (a), compute $F\left(\frac{9\pi}{4}\right)$.

(c) (2 points) Use the Fourier series that you computed in part (a) to compute the sum $1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \ldots$

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