First Midterm Examination
Thursday October 2 2003
Closed Books and Closed Notes
All Three Questions Carry Equal Points

Question 1
A Projectile Problem

A baseball is thrown with an initial velocity $\mathbf{v}_0 = v_{0x}\mathbf{E}_x + v_{0y}\mathbf{E}_y$ from the origin O (see Figure 1). The trajectory of the center of the baseball is found to satisfy the following equation:

$$\mathbf{r} = (v_{0x}t)\,\mathbf{E}_x + \left(v_{0y}t - \frac{1}{2}gt^2\right)\mathbf{E}_y. \tag{1}$$

In this problem, we ignore any rotational motion of the baseball and model it as a particle of mass m.

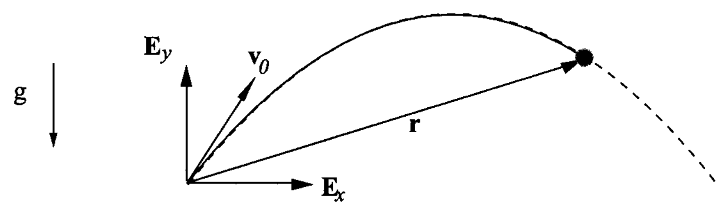


Figure 1: Schematic of the path of a baseball tossed into the air.

(a) Derive an expression for the path of the baseball using a cylindrical polar coordinate system (r, θ, z).

(b) Show that the acceleration vector of the baseball is $-g\sin(\theta)\mathbf{e}_r - g\cos(\theta)\mathbf{e}_\theta$.

(c) If $v_{0x} \neq 0$, then show that the path of the baseball can be considered as a parabola:

$$y = f(x) = c_1x + c_2x^2, \tag{2}$$

where c_1 and c_2 are constants.

(d) What is the unit tangent vector $\mathbf{e}_t$ to the path of the baseball as a function of x? What is the speed v of the baseball as a function of x and $\dot{x}$?

Question 2

A Vehicle Dynamics Problem

A vehicle moves on race track. Five times each second, the position of the car is determined using a GPS receiver. Using this information the automotive engineer is able to determine the position vector of the center of mass of the vehicle:

$$\mathbf{r} = \mathbf{r}(s(t)). \tag{3}$$

By differentiating this data, the engineer also determines

$$\mathbf{v} = \mathbf{v}(s(t)), \quad \mathbf{a} = \mathbf{a}(s(t)). \tag{4}$$

Modeling the vehicle as a particle of mass m, we assume that, in addition to a gravitational force $-mg\mathbf{E}_z$, an aerodynamic drag force $\mathbf{F}_d$ acts on the vehicle:

$$\mathbf{F}_d = -mc_d C v^2 \frac{\mathbf{v}}{v}. \tag{5}$$

Here, c_d and C are constants.

(a) Show how the data that the engineer possesses for $\mathbf{v}$ and $\mathbf{a}$ can be used to determine $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_b\}$, and the radius of curvature ρ of the path of the center of mass of the vehicle.

(b) Draw a freebody diagram of the vehicle.

(c) Using the balance of linear momentum, what is the equation governing $\dot{v}$ of the vehicle?

(d) Again using a balance of linear momentum, show that the normal force acting on the vehicle is

$$\mathbf{N} = \left(\frac{mv^2}{\rho} + mg\mathbf{E}_z \cdot \mathbf{e}_n\right)\mathbf{e}_n. \tag{6}$$

(e) The engineer's boss proposes that the friction (or traction) force on the vehicle is dynamic Coulomb friction. Using a balance of linear momentum, how could the engineer verify his boss's proposal.

Question 3

A Crate on a Spinning Disk

As shown in Figure 2, a crate of mass m is placed on a rough disc which is spinning about the vertical axis with an angular speed $\Omega = \Omega(t)$. The surface of the disc is rough and has a coefficient of static friction μ_s and a coefficient of dynamic friction μ_k. The crate is attached to the center O of the disk by a linear spring of stiffness K and unstretched length L. A vertical gravitational force acts on the disk and the crate.

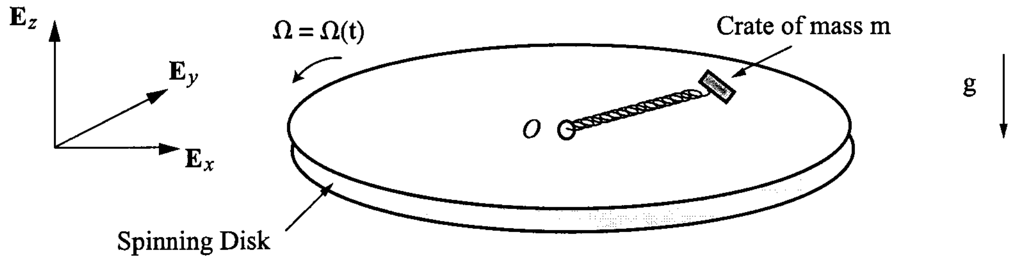


Figure 2: A crate of mass m moving on a spinning rigid disk. The center O of the disk is stationary.

(a) Modeling the crate as a particle, starting with $\mathbf{r} = r\mathbf{e}_r + 0\mathbf{E}_z$, derive expressions for $\mathbf{v}$ and $\mathbf{a}$. (In your answer clearly state any intermediate results that you use.)

(b) How do these expressions you established in **(a)** simplify if the crate is not moving relative to spinning disk? What is $\mathbf{v}_{rel}$ when the crate is moving on the surface of the disk?

(c) For the two cases where the crate is moving relative to the disk and where it is stationary relative to the disk, draw free-body diagrams of the crate.

(d) For the case where the crate is stationary relative to the disk, show that

$$\mathbf{N} = mg\mathbf{E}_z, \quad \mathbf{F}_f = \left(K(r-L) - mr\Omega^2\right)\mathbf{e}_r + mr\dot{\Omega}\mathbf{e}_\theta. \tag{7}$$

(e) Answer either (i) *or* (ii):

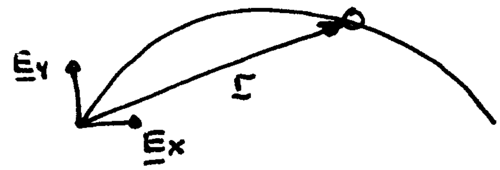
(i) Suppose that the disk is spun up from rest with a constant angular acceleration $\dot{\Omega} = \dot{\Omega}_0$. Initially, the crate is at rest on the surface of the disk. Determine the time T when the crate would commence moving on the surface of the disk.

(ii) Suppose the crate is moving on the surface the disk, using a balance of linear momentum, establish the differential equations governing the coordinates r and θ of the crate.

Problem 1

Given:

$$\underline{r} = v_{ox} t \, \underline{E}_x + \left(v_{oy} t - \tfrac{1}{2} g t^2\right) \underline{E}_y$$



(a) $\underline{r} = r \underline{e}_r$ where

$$r^2 = v_{ox}^2 t^2 + v_{oy}^2 t^2 + \tfrac{1}{4} g^2 t^4 - g v_{oy} t^3$$

$$\theta = \tan^{-1}\left(\frac{v_{oy} t - \frac{1}{2} g t^2}{v_{ox} t}\right)$$

$$\underline{e}_r = \cos\theta \, \underline{E}_x + \sin\theta \, \underline{E}_y$$

(See added comments)

(b) From $\underline{r}$ above: $\dot{\underline{r}} = \underline{v} = v_{ox} \underline{E}_x + (v_{oy} - g t) \underline{E}_y$

$$\underline{a} = \dot{\underline{v}} = 0 \, \underline{E}_x - g \underline{E}_y$$

$$= -g \left(\underline{E}_y = \sin\theta \, \underline{e}_r + \cos\theta \, \underline{e}_\theta \right)$$

$$= -g \sin\theta \, \underline{e}_r - g \cos\theta \, \underline{e}_\theta$$

Trying to establish this result from $\dot{\underline{r}} = \frac{d}{dt}(r \underline{e}_r)$ is possible but it is considerably more lengthy.

(c) From given $\underline{r}$: $x = v_{ox} t \Rightarrow t = \dfrac{x}{v_{ox}}$

$$\Rightarrow y = v_{oy} t - \tfrac{1}{2} g t^2 = \left(\frac{v_{oy}}{v_{ox}}\right) x - \frac{1}{2}\left(\frac{g}{v_{ox}^2}\right) x^2$$

Hence $y = f(x)$ is a parabola.

(d) $\underline{r} = v_{ox} t \, \underline{E}_x + \left(v_{oy} t - \tfrac{1}{2} g t^2\right) \underline{E}_y$

$\Rightarrow \underline{r}(x) = x \underline{E}_x + \left(c_1 x + c_2 x^2\right) \underline{E}_y$

$c_1 = \dfrac{v_{oy}}{v_{ox}}, \quad c_2 = -\dfrac{1}{2} \dfrac{g}{v_{ox}^2}$

$\Rightarrow \underline{v} = \dot{x} \dfrac{d\underline{r}}{dx} = \dot{x}\left(\underline{E}_x + (c_1 + 2 c_2 x) \underline{E}_y \right)$

Now $v = \|\underline{v}\| = |\dot{x}| \sqrt{1 + (c_1 + 2 c_2 x)^2}$

and $\underline{e}_t = \dfrac{\underline{v}}{v} = \dfrac{\dot{x}}{|\dot{x}|} \dfrac{1}{\sqrt{1 + (c_1 + 2 c_2 x)^2}} \left[\underline{E}_x + (c_1 + 2 c_2 x) \underline{E}_y \right]$

where $\dot{x} = v_{ox}$.

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(a) Added Comments:

I. many students started with

$$\underline{r} = V_{ox} t \, \underline{e}_x + (V_{oy} t - \tfrac{1}{2} g t^2) \underline{e}_y$$

and substituted for $\underline{e}_x$ and $\underline{e}_y$ to find

$$\underline{r} = (V_{ox} t \cos\theta + (V_{oy} t - \tfrac{1}{2} g t^2) \sin\theta) \underline{e}_r$$
$$+ (V_{ox} t (-\sin\theta) + (V_{oy} t - \tfrac{1}{2} g t^2) \cos\theta) \underline{e}_\theta$$

This result is correct, However, one needs to define $\theta = \tan^{-1}(y/x)$ with this definition of θ, you see that

$$-V_{ox} t \sin\theta + (V_{oy} t - \tfrac{1}{2} g t^2) \cos\theta = 0$$

Hence

$$\underline{r} = (V_{ox} t \cos\theta + (V_{oy} t - \tfrac{1}{2} g t^2) \sin\theta) \underline{e}_r$$

If we substitute for $\cos\theta$ and $\sin\theta$ we find that this gives

$$\underline{r} = (V_{ox}^2 t^2 + V_{oy}^2 t^2 + \tfrac{1}{4} g^2 t^4 - V_{oy} g t^3)^{1/2} \underline{e}_r$$

As expected.

II Another mistake was to write $V_{ox} = V_o \cos\theta$, $V_{oy} = \sin\theta \, V_o$.

This θ is not the same as the polar coordinate θ.

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Question 2

(a) Given $\underline{r} = \underline{r}(s(t)) \Rightarrow \underline{v} = \dot{s}\underline{e}_t$ and $\underline{a} = \ddot{s}\underline{e}_t + KV^2\underline{e}_n$

where $v = \dot{s}$.

Given $\underline{v}$, by dividing $\underline{v}$ by its magnitude $\dot{s}$, we can determine $\underline{e}_t$.

Differentiating $\dot{s}$, gives $\ddot{s}$, and so we can calculate

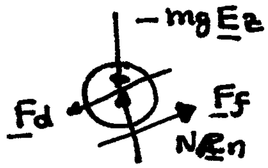
$$K\underline{e}_n = (\underline{a} - \ddot{s}\underline{e}_t)/V^2$$

From this equation we can calculate K and $\underline{e}_n$:

$$K = \| (\underline{a} - \ddot{s}\underline{e}_t)/V^2 \| \quad , \quad \underline{e}_n = \frac{\underline{a} - \ddot{s}\underline{e}_t}{\|\underline{a} - \ddot{s}\underline{e}_t\|}$$

The radius of curvature $\rho = 1/K$, and $\underline{e}_b = \underline{e}_t \times \underline{e}_n$.

(b)



$$\underline{F}_f = F_{ft}\underline{e}_t + F_{fb}\underline{e}_b$$

$$\underline{F}_d = -C_d m C V^2 \underline{e}_t$$

(c) $\underline{F} = m\underline{a} : \cdot \underline{e}_t \qquad m\dot{v} = \underline{F}\cdot\underline{e}_t$

$$= F_{ft} - C_d m C V^2 - mg\underline{E}_z\cdot\underline{e}_t$$

$$\Rightarrow \quad \dot{v} = \frac{1}{m}(F_{ft}) - C_d C V^2 - g(\underline{E}_z\cdot\underline{e}_t)$$

(d) $(\underline{F} = m\underline{a}) : \cdot\underline{e}_n : \quad mv^2/\rho = N - mg\underline{E}_z\cdot\underline{e}_n + (\underline{F}_f + \underline{F}_d)\cdot\underline{e}_n$

$$= N - mg\underline{E}_z\cdot\underline{e}_n$$

Solving for N: $\quad \underline{N} = N\underline{e}_n = (mv^2/\rho + mg\underline{E}_z\cdot\underline{e}_n)\underline{e}_n$

(e) If the friction were Coulomb: $\underline{F}_f = -\mu_K \|\underline{N}\| \underline{e}_t$

From (d) we know $\|\underline{N}\|$, and so we substitute for the proposed $\underline{F}_f$ into the differential equation from (c):

$$m(\dot{v} + C_d C V^2 - g(\underline{E}_z\cdot\underline{e}_t)) = F_{ft}. \qquad (*)$$

If the Boss is right, then the lhs of this equation should equal

$$-\mu_K \| mv^2/\rho + mg\underline{E}_z\cdot\underline{e}_n \|$$

From (a) the engineer can determine

$$\dot{v},\ v,\ \underline{E}_z \cdot \underline{e}_t$$

so the left-hand side of (*) is known. He then needs to find a μ_K such that the left-hand side matches

$$-\mu_K \| mv^2/\rho + mg\,\underline{E}_z \cdot \underline{e}_n \|$$

This μ_K needs to be constant.

Alternatively, the engineer needs to show that $F_{fb} \neq 0$. To do this s|he needs to look at $(\underline{F} = m\underline{a}) \cdot \underline{e}_b$:

$$F_{fb} = mg\,\underline{E}_z \cdot \underline{e}_b$$

From (a) s|he can find $\underline{e}_b \cdot \underline{E}_z$, so if $\underline{e}_b \cdot \underline{E}_z \neq 0$ for any portions of the vehicles path, then the friction force cannot be Coulomb

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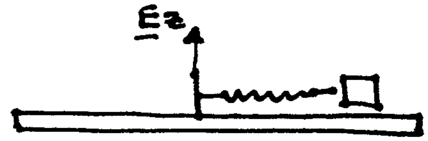
Question 3



(a) $\underline{r} = r\underline{e}_r \Rightarrow \underline{v} = \dot{r}\underline{e}_r + r\dot{\theta}\underline{e}_\theta$

$\Rightarrow \underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$

where we used $\dot{\underline{e}}_r = \dot{\theta}\underline{e}_\theta$, $\dot{\underline{e}}_\theta = -\dot{\theta}\underline{e}_r$

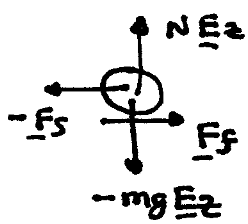
(b) If the particle is stationary relative to the dish: $\dot{r} = 0$ and $\dot{\theta} = \Omega$:

$$\underline{v} = r\Omega\underline{e}_\theta, \quad \underline{a} = -r\Omega^2\underline{e}_r + r\dot{\Omega}\underline{e}_\theta$$

The relative velocity vector $\underline{v}_{rel} = \underline{v} - r\Omega\underline{e}_\theta$

$= \dot{r}\underline{e}_r + r(\dot{\theta} - \Omega)\underline{e}_\theta$

(c)



Static: $\underline{F}_f = F_{fr}\underline{e}_r + F_{f\theta}\underline{e}_\theta$

Dynamic: $\underline{F}_f = -\mu_K \|\underline{N}\| \dfrac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|}$

Spring force: $\underline{F}_s = -K(r-L)\underline{e}_r$

(d) When particle doesn't move relative to the dish: friction is static

$$m\underline{a} = \underline{F} \Rightarrow \cdot\underline{e}_r \quad -mr\Omega^2 = F_{fr} - K(r-L)$$

$$\cdot\underline{e}_\theta \quad mr\dot{\Omega} = F_{f\theta}$$

$$\cdot\underline{E}_z \quad N = mg$$

Hence $\underline{N} = mg\underline{E}_z$, $\underline{F}_f = (-mr\Omega^2 + K(r-L))\underline{e}_r + mr\dot{\Omega}\underline{e}_\theta$

(e) (i) The particle starts moving relative to the dish when

$$\|\underline{F}_f\| = \mu_s\|\underline{N}\| = \mu_s mg$$

When dish starts from rest and $\dot{\Omega} = \dot{\Omega}_0$, $\Omega = \dot{\Omega}_0 t$, also r is constant so

$$\|\underline{F}_f\| = \mu_s mg$$

gives an equation for T (the time the particle ceases to be stuck).

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Hence, time to start moving is found by solving the following equation for T

$$(\mu_s mg)^2 = \left(-mr(\dot{\Omega}_0 T) + K(r-L)\right)^2 + m^2 r^2 \dot{\Omega}_0^2$$

where r is a constant. Solving this equation:

$$T = \frac{-m\left(\mu_s^2 g^2 - r^2\dot{\Omega}_0^2\right)^{1/2} + K(r-L)}{mr\dot{\Omega}_0}$$

With some rearranging:

$$T = \frac{K}{mr\dot{\Omega}_0}(r-L) - \left(\frac{\mu_s^2 g^2}{r^2\dot{\Omega}_0^2} - 1\right)^{1/2}$$

(ii) When the crate moves relative to the disk, the friction force is dynamic

From $\underline{F} = m\underline{a}$ where

$$\underset{\sim}{F} = -K(r-L)\underline{e}_r + (N-mg)\underline{E}_z - \mu_K |N| \frac{\underline{v}_{rel}}{\|\underline{v}_{rel}\|}$$

we find

$$\cdot\,\underline{e}_r : \quad m(\ddot{r} - r\dot{\theta}^2) = -K(r-L) - \mu_K |N| \frac{\underline{v}_{rel}\cdot\underline{e}_r}{\|\underline{v}_{rel}\|}$$

$$\cdot\,\underline{e}_\theta : \quad m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\mu_K |N| \frac{\underline{v}_{rel}\cdot\underline{e}_\theta}{\|\underline{v}_{rel}\|}$$

$$\cdot\,\underline{E}_z : \quad -mg + N = 0$$

Hence, the differential equations are

$$m\ddot{r} - mr\dot{\theta}^2 = -K(r-L) - \mu_K mg \frac{\dot{r}}{\|\underline{v}_{rel}\|}$$

$$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = -\mu_K mg \frac{r(\dot{\theta} - \Omega)}{\|\underline{v}_{rel}\|}$$

where $\underline{v}_{rel}$ is given in (b).