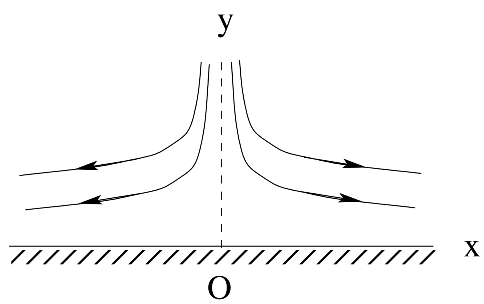
UNIVERSITY OF CALIFORNIA, BERKELEY
MECHANICAL ENGINEERING
ME106 Fluid Mechanics **NAME** ______________________________
1st Test, S18 *Prof S. Morris*

1. (100) The expression $\mathbf{V} = kxy\,\mathbf{i} - \frac{1}{2}ky^2\,\mathbf{j}$ (constant $k > 0$) represents the velocity field near a stagnation point; the no–slip condition $v_x = 0$ is satisfied on the rigid surface $y = 0$.



(a) Find the equation of the streamline passing through the point $(1, 1)$ (35 points).

(b) Find the fluid acceleration $\mathbf{a}$ (35 points).

(c) Verify that your answer in part (b) is dimensionally correct (10 points).

(d) On the figure above, sketch a fluid particle, its position vector $\mathbf{r}$ and its acceleration $\mathbf{a}$. Then explain physically the direction of the vector $\mathbf{a}$ (10 points).

(e) Explain physically why the fluid acceleration is non–zero, even though the expression for $\mathbf{V}$ is independent of time (10 points).

Solution

(a) 35 Substitute $v_x = kxy$, $v_y = -\frac{1}{2}ky^2$ into $\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y}$ (15 points - correct statement of ODE):

$$\frac{\mathrm{d}x}{xy} = -\frac{\mathrm{d}y}{\frac{1}{2}y^2} \tag{1.1}$$

(k cancels).

Rearrange :

$$\frac{1}{2}y^2\mathrm{d}\,x + xy\,\mathrm{d}\,y = 0 \ \ \Rightarrow \mathrm{d}\{xy^2\} = 0 \tag{1.2a, b}$$

Eq.(1.2b) follows from (1.2a) by the product rule. One could also cancel the common factor of y from (1.2a); the resulting equation is separated, and can be integrated.

Streamlines: xy^2 =const. (15 points - answer attained with correct method)

Streamline through $\{1, 1\}$: substitute $x = 1$, $y = 1$ into xy^2 =const. to show that const.=1, and that $xy^2 = 1$ (5 points - evaluated at (1,1) for constant).

(b) 35 Method 1:

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + v_x\frac{\partial \mathbf{V}}{\partial x} + v_y\frac{\partial \mathbf{V}}{\partial y} \tag{1.3}$$

Here $\mathbf{V} = kxy\mathbf{i} - \frac{1}{2}ky^2\mathbf{j}$:

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \ \ \frac{\partial \mathbf{V}}{\partial x} = ky\mathbf{i}, \ \ \frac{\partial \mathbf{V}}{\partial y} = kx\mathbf{i} - ky\mathbf{j}. \tag{1.4a, b, c}$$

Hence

$$\mathbf{a} = kxy\{ky\mathbf{i}\} - \frac{1}{2}ky^2\{kx\mathbf{i} - ky\mathbf{j}\}, = \frac{1}{2}k^2y^2\{x\mathbf{i} + y\mathbf{j}\}, = \frac{1}{2}k^2y^2\mathbf{r} \qquad (1.5a, b, c)$$

Method 2: because the unit vectors **i**, **j** for Cartesian coordinates are constant in direction (as well, of course, in magnitude):

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{i}\frac{\mathrm{d}v_x}{\mathrm{d}t} + \mathbf{j}\frac{\mathrm{d}v_y}{\mathrm{d}t} \qquad (1.6)$$

Because $v_x = kxy$,

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = ky\frac{\mathrm{d}x}{\mathrm{d}t} + kx\frac{\mathrm{d}y}{\mathrm{d}t}, = kyv_x + kxv_y, = \frac{1}{2}kxy^2. \qquad (1.7a, b, c)$$

Similarly,

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{1}{2}ky^3. \qquad (1.8)$$

By substituting (1.8) and (1.7c) into (1.60, we again obtain (1.5c).

15 points - equation showing acceleration as material derivative of velocity

20 points - evaluation of **a**

(-5 points for minor algebraic error)

(c) 10 For the formula for **V** to be dimensionally consistent, $[k] = L^{-1}T^{-1}$.

Dimensions of the right side of (1.5c):

$$[k^2y^2\mathbf{r}] = L^{-2}T^{-2}L^2\,L, = LT^{-2}$$

eq.(1.5c) is therefore dimensionally consistent.

(d) 10 Sketch : this must show that the vector **a** is parallel to **r** (5 points).

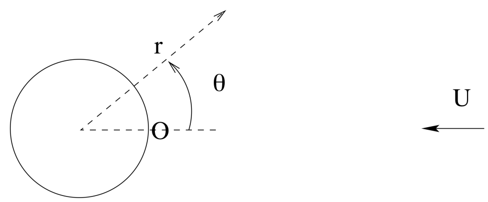
Physical interpretation: acceleration has a component directed towards the centre of curvature of the particle path. Because this flow is steady, particles move along streamlines: acceleration therefore has a component directed towards the centre of curvature of the streamline (5 points).

(e) 10 The particle velocity changes because it moves through a spatially varying velocity field. ($\frac{\partial \mathbf{V}}{\partial t} = 0$ but $v_x\frac{\partial \mathbf{V}}{\partial x} + v_y\frac{\partial \mathbf{V}}{\partial y} \neq 0$.)

2. (100) The velocity field in high–Reynolds number flow past a spherical gas bubble of radius a is given to a good approximation by

$$v_r = -U\left\{1 - \frac{a^3}{r^3}\right\}\cos\theta, \qquad v_\theta = U\left\{1 + \frac{1}{2}\frac{a^3}{r^3}\right\}\sin\theta. \tag{2.1a, b}$$

The flow is axisymmetric about the line $\theta = 0$. The flow is steady, adiabatic and effectively incompressible and inviscid.



(a) Using the Bernoulli equation, find the pressure p at point θ on the sphere ($r = a$) in terms of the fluid density ρ, pressure p_0 at the stagnation point O, and the velocity U of the free stream (that is, the fluid at infinity). Given: you may set $\mathbf{g} = 0$ (35 points).

(b) Sketch the relation between $(p - p_0)/(\frac{1}{2}\rho U^2)$ and θ calculated in part (a), and explain its form physically (35 points).

(c) Hence find the component of force F exerted by the fluid on the spherical bubble in the direction of the free stream. Using your sketch in part (b), explain your result physically (30 points).

<u>Solution</u>

(a) 35 Apply the Bernoulli equation along the stagnation streamline from O to θ (20 points):

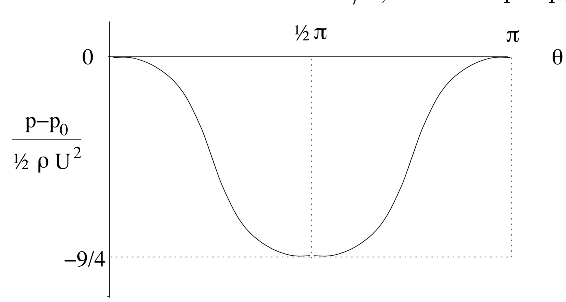
$$p + \frac{1}{2}\rho v_\theta^2 = p_0. \tag{2.2}$$

(On $r = a$, $V^2 = v_r^2 + v_\theta^2, = v_\theta^2$ because $v_r = 0$.)

Use (2.1b) to evaluate $v_\theta = \frac{3}{2}U\sin\theta$. Then substitute in (2.2) and rearrange (15 points):

$$\frac{p - p_0}{\frac{1}{2}\rho U^2} = -\frac{9}{4}\sin^2\theta \tag{2.3}$$

(b) 35 Because the speed V is an even function of $\theta - \pi/2$, so too is $p - p_0$ (15 points for explanation):



Sketch was negative, $\frac{p-p_0}{\frac{1}{2}\rho U^2}$ between 0 and 9/4, θ symmetric about $\pi/2$, zero slope at zeros (20 points).

(c) 30 The force per unit area exerted in the direction of the free stream is $-p\cos\theta$. Because this quantity is independent of azimuthal angle, the resultant force on an strip subtending angle $\mathrm{d}\theta$ at the centre of the sphere is $-p\cos\theta\,\mathrm{d}A$ (20 points), where $\mathrm{d}A = (a\,\mathrm{d}\theta)(2\pi a\sin\theta)$ (5 points). The resultant force on the entire sphere

$$F = -\int p\cos\theta\,\mathrm{d}A, = -2\pi a^2\int_0^\pi p\sin\theta\cos\theta\,\mathrm{d}\theta. \tag{2.4a,b}$$

Because a constant pressure p_0 exerts zero resultant force on a closed surface, (2.4b) can be expressed as

$$F = -2\pi a^2\int_0^\pi (p-p_0)\sin\theta\cos\theta\,\mathrm{d}\theta, \tag{2.5}$$

(Mathematically, $\int_0^\pi \sin\theta\cos\theta\,\mathrm{d}\theta = 0$ because, with respect to $\theta - \pi/2$, $\sin\theta$ is an even function, but $\cos\theta$ is an odd function.)

Because, with respect to $\theta - \pi/2$, $(p-p_0)\sin\theta$ is an even function, but $\cos\theta$ is an odd function, the integral vanishes: F = 0. This result can also be deduced by expressing (2.4a) in the form

$$F = -\int (p-p_0)\cos\theta\,\mathrm{d}A,$$

and noting that $\mathrm{d}A > 0$, but $(p-p_0)\cos\theta$ is an odd function of $\theta - \pi/2$ (5 points for explanation).

(-5 points if body was assumed a cylinder instead of a sphere)

END